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New Instanton Effects in String Theory

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We describe a new class of instanton effects in string compactifications that preserve only $\mathcal{N} = 1$ supersymmetry in four dimensions. As is well-known, worldsheet or brane instantons in such a background can sometimes contribute to an effective superpotential for the moduli of the compactification. We generalize this phenomenon by showing that such instantons can also contribute to new multi-fermion and higher-derivative F -terms in the low-energy effective action. We consider in most detail the example of heterotic compactification on a Calabi-Yau threefold X with gauge bundle V , in which case we study worldsheet instanton effects that deform the complex structure of the moduli space associated to X and V . We also give new, slightly more economical derivations of some previous results about worldsheet instantons in Type IIA Calabi-Yau compactifications.

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1. Introduction

An important non-perturbative feature of string theory or M -theory compactifications which preserve only $\mathcal{N} = 1$ supersymmetry in four dimensions is that fundamental string worldsheet or brane instantons can sometimes generate an effective superpotential for the light chiral superfields that describe the classical moduli of the background. Such a superpotential drastically alters the low-energy behavior of the theory, since some or all of the branches of the classical moduli space can be lifted. As a result, these instanton effects play a prominent role in recent attempts to construct four-dimensional string vacua with no moduli (see for instance [1] and references therein).

In practice, an explicit and rigorous computation of the instanton-generated superpotential is quite hard. The superpotential contribution from each instanton typically involves a one-loop functional determinant whose moduli dependence must be analyzed (see the work of Buchbinder and collaborators [2,3] for a beautiful example of such analysis), and the results must then be summed over what may be a myriad of contributing instantons. Furthermore, even if the individual instanton contribution is generically non-zero, the instanton sum can still vanish, as for worldsheet instantons in heterotic Calabi-Yau compactifications described by $(0, 2)$ linear sigma models [4–6].

However, one heuristic reason to believe that string worldsheet and brane instantons often generate a superpotential is that an analogous phenomenon already occurs in the much simpler context of four-dimensional supersymmetric gauge theory. As shown by Affleck, Dine, and Seiberg [7], instantons in supersymmetric QCD (or SQCD) with gauge group $SU(N_c)$ and with $N_f = N_c - 1$ flavors² generate a superpotential that completely lifts the classical moduli space of supersymmetric vacua of the theory.

Besides providing a sterling example of an instanton-generated superpotential, SQCD also provides an example of a class of more subtle instanton effects whose stringy analogues have not been much considered. The most prominent such effect occurs in SQCD with $N_f = N_c$ flavors. As shown by Seiberg [8] (related issues were also discussed in [9]), instantons in this theory do not generate a superpotential, but they nonetheless deform the complex structure of the classical moduli space. This quantum deformation is not so drastic as to lift any branches of the classical moduli space, but it instead smooths away a classical singularity at the origin of moduli space.

² We recall that a flavor is a massless chiral multiplet transforming in the sum of the fundamental and the anti-fundamental representations of the gauge group.

The exotic instanton effect in SQCD with $N_f = N_c$ flavors raises an immediate question about analogous phenomena in string theory. Can worldsheet or brane instantons which do *not* generate a superpotential nonetheless generate a quantum deformation of the moduli space? If so, what form can this deformation take?

The basic purpose of this paper is to show that such new, intrinsically stringy instanton effects can occur and to explain how they can be systematically computed.

A Brief Sketch of the Main Idea

In any instanton computation, an initial and very important step is to determine what sort of term in the low-energy effective action can possibly be generated by the instanton. Thus, if the instanton is to contribute to the superpotential and hence is to generate an F -term correction of the usual form

$$\delta S = \int d^4x d^2\theta W(\Phi^i), \quad (1.1)$$

where the Φ^i are light chiral superfields which locally parameterize the moduli space, then the instanton must carry two fermion zero-modes to produce the fermionic chiral measure $d^2\theta$ in δS . If there are precisely two fermion zero-modes in the instanton background, a superpotential is generated straightforwardly. If there are more than two such zero-modes, a superpotential cannot be generated unless some higher-order interactions lift the extra zero-modes, which indeed occurs in the example of SQCD with $N_f = N_c - 1$ flavors.

On the other hand, as we discussed in the context of SQCD in [10] and as we review here in Section 2, a different sort of F -term in the effective action is needed to describe intrinsically a complex structure deformation of the moduli space. Schematically, this F -term is a multi-fermion F -term of the form

$$\delta S = \int d^4x d^2\theta \omega_{\bar{i}\bar{j}} \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{i}} \overline{D}^{\dot{\alpha}} \overline{\Phi}^{\bar{j}}, \quad (1.2)$$

where $\overline{D}_{\dot{\alpha}}$ is the usual spinor covariant derivative on superspace and ω is a tensor on the moduli space that represents the deformation. Since this F -term involves a pair of the fermionic superfields $\overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{i}}$, it leads in components to an effective four-fermion vertex, and hence it can be generated most directly by an instanton that carries four fermion zero-modes, as does the instanton in SQCD with $N_f = N_c$ flavors once the effects of higher-order interactions are included.

More generally, we are free to consider multi-fermion F -terms involving an arbitrary number of fermions. As shown in [10], these F -terms are also generated in SQCD in

the parameter regime $N_f > N_c$, for which the instantons carry $2(N_f - N_c + 2)$ fermion zero-modes. Hence we can generalize our question above to ask if worldsheet or brane instantons possibly generate the higher multi-fermion F -terms as well.

As our discussion makes clear, if we seek worldsheet or brane instantons which straightforwardly generate more exotic F -terms than the superpotential, we should simply look for supersymmetric instanton configurations that carry extra fermion zero-modes. To describe the examples that we primarily study in this paper, we consider perturbative heterotic compactification on a Calabi-Yau threefold X with a stable, holomorphic gauge bundle V over X . In this context, the instantons we study are supersymmetric worldsheet instantons that wrap smooth holomorphic curves embedded in X .

We recall from [11] that if C is such a holomorphic curve, then C can straightforwardly generate a superpotential only if C has genus zero (so that C is a rational curve) and only if C is isolated as a holomorphic curve in X . This result follows directly from the counting of fermion zero-modes on C , since if C has higher genus or moves in a holomorphic family of curves in X , then C carries additional fermion zero-modes beyond the two that are needed to generate a superpotential. For embedded curves of genus zero that move in a family in X , higher-order interactions potentially lift the additional fermion zero-modes and enable the generation of a superpotential, but we have been unable to see how this can occur or to prove conclusively that it does not. Curves of higher genus do not contribute to the superpotential, since a familiar argument based on spacetime supersymmetry shows that contributions to the superpotential only arise at tree-level in string perturbation theory.

Conversely, curves of higher genus or curves that move in families in X do straightforwardly generate higher-order F -term interactions. This phenomenon is what we explore in the present paper, and as a particularly interesting case, we show that the curves which most directly generate the F -term in (1.2) describing a complex structure deformation of the moduli space are rational curves that move in a one-parameter family in X .

The Plan of the Paper

The plan for the paper is as follows. First, in Section 2 we recall from [10] a few general facts about multi-fermion F -terms and in particular how they can describe a complex structure deformation of the classical moduli space.

Next, in Section 3 we review following [12] how to perform worldsheet instanton computations in the physical gauge, or Green-Schwarz, formalism. This formalism makes the counting of physical fermion zero-modes on C transparent and hence is particularly suited to our computation of multi-fermion F -terms.

In Section 4 we apply the physical gauge formalism to compute the F -term generated by a one-parameter family of rational curves in X . We show that such a family can deform the complex structure of the moduli space, and we describe the general features of this deformation.

Finally, in Section 5 we generalize the computation in Section 4 to show that additional higher-derivative and multi-fermion F -terms are generated by higher genus holomorphic curves and curves that move in multi-parameter families in X . In the process, we also use the physical gauge formalism to give concise derivations of some older results about related F -terms generated by closed string [13–15] and open string [16–20] worldsheet instantons in Type IIA Calabi-Yau compactifications.

The present paper has some overlap with the work of Antoniadis and collaborators [21–23], who also consider multi-fermion and higher-derivative F -terms as generated by worldsheet instantons in heterotic and Type IIA compactifications. In their work, these exotic F -terms are primarily studied from the perspective of holomorphic anomalies [13,24] in the 1PI-effective action. In other words, they study F -terms that appear in the 1PI effective action after integrating out massless modes even if no such F -terms are present in the underlying Wilsonian effective action. Our purpose is to study the F -terms in the Wilsonian effective action.

2. General Remarks on Multi-Fermion F -terms

In this section, we briefly review some observations from [10] about multi-fermion F -terms in four-dimensional, $\mathcal{N} = 1$ supersymmetric effective actions. For simplicity, we consider only theories with global (as opposed to local) supersymmetry. In the context of our study of $\mathcal{N} = 1$ supersymmetric string backgrounds, we ultimately achieve the necessary decoupling of four-dimensional gravity by working with non-compact, local models for the Calabi-Yau threefold X . The technical extension of our work to include the coupling to four-dimensional gravity might be of interest, but we do not consider it here.

2.1. A Multi-Fermion F -term For a Deformation of Complex Structure

To motivate our study of multi-fermion F -terms in the effective action, let us begin by considering how to describe physically the infinitesimal deformation of some classical moduli space \mathcal{M}_{cl} to a quantum moduli space \mathcal{M} . Abstractly, the classical effective action associated to motion on \mathcal{M}_{cl} is a four-dimensional, $\mathcal{N} = 1$ supersymmetric nonlinear sigma

model which describes maps $\Phi : \mathbb{R}^{4|4} \longrightarrow \mathcal{M}_{cl}$. This sigma model is governed by the usual action,

$$S = \int d^4x d^4\theta K(\Phi^i, \bar{\Phi}^{\bar{i}}) . \quad (2.1)$$

Here Φ^i and $\bar{\Phi}^{\bar{i}}$ are respectively chiral and anti-chiral superfields whose lowest bosonic components describe local holomorphic and anti-holomorphic coordinates on \mathcal{M}_{cl} , and K is the Kahler potential associated to some Kahler metric $ds^2 = g_{i\bar{i}} d\phi^i d\bar{\phi}^{\bar{i}}$ on \mathcal{M}_{cl} .

Similarly, the quantum effective action is also a nonlinear sigma model as above, but now with target space \mathcal{M} instead of \mathcal{M}_{cl} . In principle, to pass from the sigma model with target \mathcal{M}_{cl} to target \mathcal{M} , we must add a correction term δS to the classical effective action. So we ask — what form does δS take?

In general, a deformation of the complex structure on \mathcal{M}_{cl} can be described intrinsically as a change in the $\bar{\partial}$ operator on \mathcal{M}_{cl} of the form

$$\bar{\partial}_{\bar{j}} \longmapsto \bar{\partial}_{\bar{j}} + \omega_{\bar{j}}^i \partial_i . \quad (2.2)$$

Here $\omega_{\bar{j}}^i$ is a representative of a Dolbeault cohomology class in $H_{\bar{\partial}}^1(\mathcal{M}_{cl}, T\mathcal{M}_{cl})$, whose elements parametrize infinitesimal deformations of \mathcal{M}_{cl} . We use standard notation, with $T\mathcal{M}_{cl}$ and $\Omega_{\mathcal{M}_{cl}}^1$ denoting the holomorphic tangent and cotangent bundles of \mathcal{M}_{cl} .

We can equally well represent the change (2.2) in the $\bar{\partial}$ operator on \mathcal{M}_{cl} as a change in the dual basis of holomorphic one-forms $d\phi^i$,

$$d\phi^i \longmapsto d\phi^i - \omega_{\bar{j}}^i d\bar{\phi}^{\bar{j}} . \quad (2.3)$$

As a result, under the deformation the metric on \mathcal{M}_{cl} changes as

$$g_{i\bar{i}} d\phi^i d\bar{\phi}^{\bar{i}} \longmapsto g_{i\bar{i}} \left(d\phi^i - \omega_{\bar{j}}^i d\bar{\phi}^{\bar{j}} \right) d\bar{\phi}^{\bar{i}} , \quad (2.4)$$

so that the metric picks up a component of type $(0, 2)$ when written in the original holomorphic and anti-holomorphic coordinates. (Of course, there is also a complex conjugate term of type $(2, 0)$ which we suppress.)

Since we know how the metric on \mathcal{M}_{cl} changes when \mathcal{M}_{cl} is deformed, we can immediately deduce the correction δS to the classical sigma model action. This correction takes the form

$$\delta S = \int d^4x d^2\theta \omega_{i\bar{j}} \bar{D}_{\dot{\alpha}} \bar{\Phi}^{\bar{i}} \bar{D}^{\dot{\alpha}} \bar{\Phi}^{\bar{j}} + c.c. = \int d^4x \omega_{i\bar{j}} d\bar{\phi}^{\bar{i}} d\bar{\phi}^{\bar{j}} + \dots , \quad (2.5)$$

with

$$\omega_{\bar{i}\bar{j}} = \frac{1}{2} \left(g_{i\bar{i}} \omega_{\bar{j}}^i + g_{i\bar{j}} \omega_{\bar{i}}^i \right). \quad (2.6)$$

Here $\overline{D}_{\dot{\alpha}}$ is the usual spinor covariant derivative on superspace. We have also performed the fermionic integral with respect to $d^2\theta$ in (2.5), from which we see that the leading bosonic term reproduces the correction to the metric in (2.4). Other components of this F -term (indicated by the ‘…’ above) include a four-fermion, non-derivative interaction from which the multi-fermion F -term takes its name.

Chirality and Cohomology

The multi-fermion F -term in (2.5) that describes the complex structure deformation of the moduli space differs from the more familiar superpotential in two important ways.

First, the superpotential arises from a holomorphic function $W(\Phi^i)$ on \mathcal{M}_{cl} and hence is manifestly supersymmetric. In contrast, the multi-fermion F -term is not manifestly supersymmetric, since the corresponding operator $\mathcal{O}_\omega = \omega_{\bar{i}\bar{j}} \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{i}} \overline{D}^{\dot{\alpha}} \overline{\Phi}^{\bar{j}}$ is not manifestly chiral. Instead, the chirality of \mathcal{O}_ω (in the on-shell supersymmetry algebra of the classical sigma model) follows from the fact that the tensor $\omega_{\bar{j}}^i$ is annihilated by $\overline{\partial}$.

Another important distinction between the multi-fermion F -term in (2.5) and the superpotential is that, unlike a holomorphic function, the cohomology class in $H_{\overline{\partial}}^1(\mathcal{M}_{cl}, T\mathcal{M}_{cl})$ that actually determines the deformation is locally trivial. This fact implies that locally on \mathcal{M}_{cl} , the multi-fermion F -term δS can be integrated to a D -term, having the form $\int d^4\theta (\dots)$. However, because the cohomology class represented by ω is globally non-trivial, we cannot write the correction δS globally on \mathcal{M}_{cl} as a D -term, and in this sense δS is an F -term.

2.2. Multi-Fermion F -terms of Higher Degree

The multi-fermion F -term in (2.5) that describes a deformation of the moduli space is only the first in a series of multi-fermion F -terms that we can consider. To exhibit the generalization, we begin with a section ω of $\overline{\Omega}_{\mathcal{M}_{cl}}^p \otimes \overline{\Omega}_{\mathcal{M}_{cl}}^p$. (Were it not for the requirement of Lorentz-invariance, we could more generally start with a section of $\overline{\Omega}_{\mathcal{M}_{cl}}^p \otimes \overline{\Omega}_{\mathcal{M}_{cl}}^q$ for $p \neq q$.) Explicitly, ω is given by a tensor $\omega_{\bar{i}_1 \dots \bar{i}_p \bar{j}_1 \dots \bar{j}_p}$ that is antisymmetric in the \bar{i}_k and also in the \bar{j}_k . Given such a tensor, we construct a possible term in the effective action that generalizes what we found in (2.5):

$$\begin{aligned} \delta S &= \int d^4x d^2\theta \omega_{\bar{i}_1 \dots \bar{i}_p \bar{j}_1 \dots \bar{j}_p} \left(\overline{D}_{\dot{\alpha}_1} \overline{\Phi}^{\bar{i}_1} \overline{D}^{\dot{\alpha}_1} \overline{\Phi}^{\bar{j}_1} \right) \dots \left(\overline{D}_{\dot{\alpha}_p} \overline{\Phi}^{\bar{i}_p} \overline{D}^{\dot{\alpha}_p} \overline{\Phi}^{\bar{j}_p} \right), \\ &\equiv \int d^4x d^2\theta \mathcal{O}_\omega. \end{aligned} \quad (2.7)$$

Given the form of this operator, we can assume that ω is symmetric under the overall exchange of i 's and j 's.

As explained in [10], the general multi-fermion F -terms of degree $p > 1$ have no effect on the classical algebraic geometry of the moduli space, and for this reason our primary interest lies in the F -term of degree $p = 1$ associated to a deformation of the moduli space. Nonetheless, in Section 5 we briefly discuss worldsheet instantons that generate F -terms of higher degree, so for completeness we state some basic properties of the general multi-fermion F -term. We refer to [10] for a more extended discussion.

Chirality of \mathcal{O}_ω

As before, the operator \mathcal{O}_ω determined by the tensor ω is not manifestly chiral. In the on-shell supersymmetry algebra of the sigma model, the chirality condition on \mathcal{O}_ω can be expressed geometrically in terms of ω as follows. We first note that we can use the Kahler metric $g_{i\bar{i}}$ on \mathcal{M}_{cl} to raise either set of \bar{i} or \bar{j} indices on ω . The raised indices become of type $(1,0)$, so upon raising the indices, ω becomes interpreted as a section of $\overline{\Omega}_{\mathcal{M}_{cl}}^p \otimes \Lambda^p T\mathcal{M}_{cl}$ in two distinct ways. By our assumption on the symmetry of ω , we find the same section of $\overline{\Omega}_{\mathcal{M}_{cl}}^p \otimes \Lambda^p T\mathcal{M}_{cl}$ either way.

We now consider the action of the anti-chiral supercharges $\overline{Q}_{\dot{\alpha}}$ in the on-shell supersymmetry algebra of the unperturbed sigma model, so that we consider for simplicity only the linearized supersymmetry constraint on δS . Under the action of $\overline{Q}_{\dot{\alpha}}$, the component fields ϕ^i and ψ_β^i of Φ^i and the component fields $\overline{\phi}^{\bar{i}}$ and $\overline{\psi}_{\dot{\beta}}^{\bar{i}}$ of $\overline{\Phi}^{\bar{i}}$ transform as

$$\begin{aligned}\delta_{\dot{\alpha}}\phi^i &= 0, & \delta_{\dot{\alpha}}\overline{\phi}^{\bar{i}} &= \overline{\psi}_{\dot{\alpha}}^{\bar{i}}, \\ \delta_{\dot{\alpha}}\psi_\beta^i &= i\partial_{\dot{\alpha}\beta}\phi^i, & \delta_{\dot{\alpha}}\overline{\psi}_{\dot{\beta}}^{\bar{i}} &= -\Gamma_{\bar{j}\bar{k}}^{\bar{i}}\overline{\psi}_{\dot{\alpha}}^{\bar{j}}\overline{\psi}_{\dot{\beta}}^{\bar{k}}.\end{aligned}\tag{2.8}$$

Here Γ is the connection associated to the Kahler metric $g_{i\bar{i}}$ on \mathcal{M} . So long as we consider only the action of a single supercharge, we can without loss set Γ to zero by a suitable coordinate choice on \mathcal{M} .

By using the metric to interpret each set of anti-chiral fermions $\overline{\psi}_{\dot{\beta}}^{\bar{i}}$ for $\dot{\beta} = 1, 2$ as alternatively anti-holomorphic one-forms $d\overline{\phi}^{\bar{i}}$ or holomorphic tangent vectors $\partial/\partial\phi^i$, we see directly from (2.8) that the action of each of the two supercharges $\overline{Q}_{\dot{\alpha}}$ on \mathcal{O}_ω corresponds to the action of $\overline{\partial}$ on ω when ω is regarded as a section of $\overline{\Omega}_{\mathcal{M}_{cl}}^p \otimes \Lambda^p T\mathcal{M}_{cl}$ in either of the two possible ways. Thus, the chirality constraint on \mathcal{O}_ω is simply the condition that ω be annihilated by $\overline{\partial}$.

Cohomology of \mathcal{O}_ω

Similar to the F -term of degree $p = 1$, the general multi-fermion F -term is really defined as a cohomology class on \mathcal{M}_{cl} . The reduction to cohomology arises because the F -term is only defined up to the addition of interactions which have the same form and which can be written as integrals over three-quarters of superspace. Such interactions necessarily appear as

$$\begin{aligned} \delta S &= \int d^4x d^2\theta d\bar{\theta}_{\dot{\alpha}_1} \xi_{\bar{i}_2 \dots \bar{i}_p \bar{j}_1 \dots \bar{j}_p} \bar{D}^{\dot{\alpha}_1} \bar{\Phi}^{\bar{j}_1} \left(\bar{D}_{\dot{\alpha}_2} \bar{\Phi}^{\bar{i}_2} \bar{D}^{\dot{\alpha}_2} \bar{\Phi}^{\bar{j}_2} \right) \dots \left(\bar{D}_{\dot{\alpha}_p} \bar{\Phi}^{\bar{i}_p} \bar{D}^{\dot{\alpha}_p} \bar{\Phi}^{\bar{j}_p} \right), \\ &\equiv \int d^4x d^2\theta d\bar{\theta}_{\dot{\alpha}_1} \mathcal{O}_\xi^{\dot{\alpha}_1}, \\ &= \int d^4x d^2\theta \nabla_{\bar{i}_1} \xi_{\bar{i}_2 \dots \bar{i}_p \bar{j}_1 \dots \bar{j}_p} \left(\bar{D}_{\dot{\alpha}_1} \bar{\Phi}^{\bar{i}_1} \bar{D}^{\dot{\alpha}_1} \bar{\Phi}^{\bar{j}_1} \right) \dots \left(\bar{D}_{\dot{\alpha}_p} \bar{\Phi}^{\bar{i}_p} \bar{D}^{\dot{\alpha}_p} \bar{\Phi}^{\bar{j}_p} \right). \end{aligned} \tag{2.9}$$

Here ξ is a section of $\overline{\Omega}_{\mathcal{M}_{cl}}^{p-1} \otimes \overline{\Omega}_{\mathcal{M}_{cl}}^p$, and ∇ is the canonical covariant derivative defined with respect to the background Kahler metric g on \mathcal{M}_{cl} . In passing to the last expression in (2.9), we have simply performed the superspace integral over $d\bar{\theta}_{\dot{\alpha}_1}$ to produce a multi-fermion F -term of the same form as in (2.7).

Because of the possibility of such corrections to the effective action, we must impose an equivalence relation on the set of chiral operators \mathcal{O}_ω that can appear in the multi-fermion F -term. This equivalence relation is given by

$$\mathcal{O}_\omega \sim \mathcal{O}_\omega + \{\overline{Q}_{\dot{\alpha}}, \mathcal{O}_\xi^{\dot{\alpha}}\} = \mathcal{O}_{\omega + \bar{\partial}\xi}. \tag{2.10}$$

Hence only if the class of \mathcal{O}_ω is non-trivial under (2.10) is the multi-fermion F -term an honest chiral interaction.

3. Worldsheet Instanton Computations in Physical Gauge

The rest of this paper is devoted to a variety of worldsheet instanton computations, so in this section we discuss general aspects of how we perform these computations. Since the fundamental string has a number of different worldsheet descriptions — for instance using the RNS, the Green-Schwarz, or the Berkovits hybrid formalism — we have at our disposal a corresponding number of ways to perform worldsheet instanton computations. See [11], [12], and [15] for basic examples of worldsheet instanton computations performed respectively with these methods.

Throughout this paper, we restrict attention to worldsheet instantons which wrap only once³ about a smooth holomorphic curve C embedded in the Calabi-Yau threefold X . In this situation, the simplest method of computation by far is to apply the Green-Schwarz formalism, after having fixed its kappa-symmetry to eliminate unphysical worldsheet degrees of freedom. This physical gauge formalism has the great advantage that spacetime supersymmetry is manifest, so that we can directly compute F -term corrections to the effective action in superspace. Furthermore, we avoid such technical complications as the need to sum over worldsheet spin structures, as in the RNS formalism.

The physical gauge framework for worldsheet instanton computations is thus precisely analogous to the formalism for brane instanton computations introduced in the case of Type II compactification by Becker, Becker, and Strominger [25] and further elucidated in the context of M -theory membrane instanton computations by Harvey and Moore [26]. In fact, if we consider the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic string, then the physical gauge formalism has a direct correspondence to the brane formalism, since S -duality with the Type I string maps heterotic worldsheet instantons to Type I $D1$ -brane instantons.

To illustrate the use of the physical gauge formalism for worldsheet instanton computations, we now review two instanton computations originally performed in [12] and which we generalize in this paper. As a first example, we compute the instanton contribution to the superpotential when C is a smooth, isolated rational curve in X . As a second example, we consider heterotic compactification on a $K3$ surface preserving $\mathcal{N} = 2$ supersymmetry, and we compute a worldsheet instanton correction to the metric on hypermultiplet moduli space.

3.1. Preliminaries

To start, we review the structure of the worldvolume theory on the heterotic string in physical gauge. The most important aspect of this structure is the fact that the worldvolume theory on C is automatically twisted, as for any brane that wraps a supersymmetric cycle [27]. This twisting is important both because it controls the structure of physical fermion zero-modes on C and because it leads to cancellations that greatly simplify our computations.

Worldvolume Bosons

³ That is, in the language of the sigma model, we do not consider multiple covers of C .

We first describe the worldvolume bosons on C in physical gauge. These bosons describe small fluctuations about C in the ten-dimensional space $\mathbb{C}^2 \times X$, where for convenience we have chosen a complex structure on the four Euclidean directions transverse to X . Thus, the worldvolume bosons are valued in the normal bundle to C in $\mathbb{C}^2 \times X$, and we identify this normal bundle with the direct sum $\mathcal{O}^2 \oplus N$. Here $\mathcal{O}^2 \equiv \mathcal{O} \oplus \mathcal{O}$ is the trivial rank two holomorphic bundle on C , and N is the holomorphic normal bundle to C in X .

We let x^μ for $\mu = 1, 2$ denote the two complex bosons on C valued in the trivial bundle \mathcal{O}^2 . We similarly let y^m for $m = 1, 2$ denote the two complex bosons on C valued in N .

Worldvolume Fermions

We now consider the worldvolume fermions on C in physical gauge. To describe spinors on C , we introduce a right-moving spin bundle S_+ on C and a left-moving spin bundle S_- on C . By convention, the kinetic operator for a right-moving fermion on C is a ∂ operator, whereas the kinetic operator for a left-moving fermion on C is a $\bar{\partial}$ operator.

We also let $S_+(\mathcal{O}^2 \oplus N)$ denote the positive chirality spin bundle associated to the normal bundle to C in $\mathbb{C}^2 \times X$. Of course we have the decomposition

$$S_+(\mathcal{O}^2 \oplus N) = [S_+(\mathcal{O}^2) \otimes S_+(N)] \oplus [S_-(\mathcal{O}^2) \otimes S_-(N)], \quad (3.1)$$

where $S_\pm(\mathcal{O}^2)$ and $S_\pm(N)$ denote the positive and negative chirality spin bundles associated to the respective rank two holomorphic bundles on C .

In physical gauge, the right-moving worldvolume fermions on C transform *a priori* as sections of the tensor product

$$S_+ \otimes S_+(\mathcal{O}^2 \oplus N) = [S_+ \otimes S_+(\mathcal{O}^2) \otimes S_+(N)] \oplus [S_+ \otimes S_-(\mathcal{O}^2) \otimes S_-(N)]. \quad (3.2)$$

We can put this in a more convenient form by expressing the fermions in terms of differential forms rather than spinors. (Being able to do this reflects the way the theory is “twisted.”) Much as the Calabi-Yau condition on X implies that spinors on X can be identified with differential forms on X (see §15.5 of [28] for a review of this statement), the same analysis as restricted to C implies the following identifications of bundles on C ,

$$\begin{aligned} S_+ \otimes S_-(N) &= \mathcal{O} \oplus \wedge^2 N^*, \\ S_+ \otimes S_+(N) &= N^*. \end{aligned} \quad (3.3)$$

Here N^* is the conormal bundle, the dual of N .

Because the kinetic operator for the right-moving fermions is the ∂ operator, we find it convenient to consider these fermions as sections of *anti-holomorphic* as opposed to holomorphic bundles on C . This presentation makes the counting of fermion zero-modes immediate, as we count anti-holomorphic sections of anti-holomorphic bundles. Thus, we use the anti-holomorphic three-form $\bar{\Omega}$ and the background metric on X to identify sections of the holomorphic bundles $\wedge^2 N^*$ and N^* appearing in (3.3) with corresponding sections of the anti-holomorphic bundles $\bar{\Omega}_C^1$ and \bar{N} . (The Calabi-Yau condition is used to express $\wedge^2 N^*$ as $\Omega_C^1{}^* \sim \bar{\Omega}_C^1$.)

Combining (3.2), (3.3), and the identifications above, we see that the right-moving worldvolume fermions on C can be alternatively identified as sections of the bundles

$$S_+(\bar{\mathcal{O}}^2) \otimes \bar{N}, \quad S_-(\bar{\mathcal{O}}^2) \otimes \bar{\mathcal{O}}, \quad S_-(\bar{\mathcal{O}}^2) \otimes \bar{\Omega}_C^1. \quad (3.4)$$

Here $\bar{\mathcal{O}}$ is still the trivial bundle on C , but we use this notation to remind ourselves that we consider the right-moving fermions to transform as sections of anti-holomorphic bundles.

We introduce the following notation for these three sets of fermions:

$$\begin{aligned} \bar{\chi}_{\dot{\alpha}}^{\bar{m}} &\in \Gamma(C, S_+(\bar{\mathcal{O}}^2) \otimes \bar{N}), \\ \theta^\alpha &\in \Gamma(C, S_-(\bar{\mathcal{O}}^2) \otimes \bar{\mathcal{O}}), \\ \theta_z^\alpha &\in \Gamma(C, S_-(\bar{\mathcal{O}}^2) \otimes \bar{\Omega}_C^1). \end{aligned} \quad (3.5)$$

Here, $\dot{\alpha}, \alpha = 1, 2$ are respectively positive and negative chirality spinor indices on $S_\pm(\bar{\mathcal{O}}^2)$ (which we can interpret simply as trivial bundles of rank 2, transforming under rotations of \mathbb{R}^4 as spinors of positive or negative chirality), and $\bar{m} = 1, 2$ is an index for \bar{N} . Also, we let z and \bar{z} be local holomorphic and anti-holomorphic coordinates on C , so that the \bar{z} -index on θ_z^α reminds us that this fermion transforms as a $(0, 1)$ -form on C . In particular, the worldvolume fermions in (3.5) are now manifestly twisted.

To complete our description of the worldvolume theory on C , we must include the left-moving $\text{Spin}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$ current algebra coupled to the background gauge field associated to the holomorphic bundle V over X . We will assume that the $\text{Spin}(32)/\mathbb{Z}_2$ bundle has vector structure, and so can be considered as an $SO(32)$ bundle. (As explained in [12], this is the natural case for the worldsheet instanton computation.) In the Type I or $\text{Spin}(32)/\mathbb{Z}_2$ heterotic description, the current algebra can then be described by a set of thirty-two left-moving fermions that transform as sections of the bundle $S_- \otimes V|_C \equiv V_-$. The kinetic operator for these bundle fermions is the $\bar{\partial}$ operator coupled to V_- . For simplicity, we assume this fermionic description of the current algebra throughout. At the appropriate times, we will comment on the extension to the $E_8 \times E_8$ heterotic string.

3.2. Computing the Superpotential

As we now illustrate, even our very schematic description of the worldvolume theory on C suffices to compute the instanton contribution to the superpotential. We assume that C is a smooth, isolated rational curve in X . The latter assumption implies that the normal bundle N is given⁴ by $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

In this situation, the worldvolume theory on C has two complex bosonic zero-modes and two right-moving fermionic zero-modes. The bosonic zero-modes arise from the complex bosons x^μ which describe translations of C in \mathbb{C}^2 , and the right-moving fermionic zero-modes arise from the fermions θ^α . These zero-modes on C are Goldstone modes associated to the four translation symmetries and to the two supersymmetries broken by the instanton.

In general, zero-modes of the fermions $\bar{\chi}_\alpha^m$ and θ_z^α arise from anti-holomorphic sections of the respective anti-holomorphic bundles \overline{N} and $\overline{\Omega}_C^1$. Such sections are conjugate to holomorphic sections of the holomorphic bundles N and Ω_C^1 , so to count the zero-modes of these fermions we need only count holomorphic sections of N and Ω_C^1 . In the case at hand, because C is isolated as a holomorphic curve in X and N has no holomorphic sections, the fermions $\bar{\chi}_\alpha^m$ do not have zero-modes. Similarly, because C has genus zero and Ω_C^1 has no holomorphic sections, the fermions θ_z^α do not have zero-modes either.

Evaluating the contribution from C to the superpotential is now admirably direct in the physical gauge formalism. To determine the contribution from C to the low-energy effective action — which amounts to integrating out the physical degrees of freedom on C — we merely evaluate the worldvolume partition function. By standard reasons of holomorphy [11], any superpotential contribution from C cannot have a non-trivial perturbative dependence on the string tension α' , so we need only evaluate this partition function to one-loop order. Hence the superpotential contribution from C can be computed as an elementary Gaussian integral over the fluctuating, physical degrees of freedom on the worldvolume.

With our previous description of the physical degrees of freedom on C , we can perform this Gaussian integral immediately. As we will explain, the F -term contribution to the effective action from C is given formally by

$$\delta S = \int d^4x d^2\theta W_C, \quad (3.6)$$

⁴ We use the standard notation for line-bundles on \mathbb{CP}^1 . Hence $\mathcal{O}(m)$ denotes the line-bundle having degree, or first Chern class, m .

with

$$W_C = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B\right) \frac{\text{Pfaff}(\bar{\partial}_{V_-})}{(\det' \bar{\partial}_{\mathcal{O}})^2 (\det \bar{\partial}_{\mathcal{O}(-1)})^2}. \quad (3.7)$$

We will describe in turn where the different factors in this formula come from. First, from the perspective of the instanton computation, the chiral measure $d^4x d^2\theta$ on superspace in (3.6) arises from integration over the collective coordinates of the instanton which are associated to its bosonic and fermionic Goldstone modes.

The exponential factor in (3.7) is simply the exponential of the classical action. The real part of the exponential is the area $A(C)$ of a worldsheet wrapping C , and the imaginary part is the coupling to this worldsheet of the B -field. Because C is holomorphic, its area $A(C)$ depends only on the Kahler class of the metric on X , and the argument of the exponential in W_C in turn represents the complexified Kahler class of X . Although we will not dwell on this point now, the precise meaning of the “period” of B over C in the complexified Kahler class is rather subtle, as emphasized in [12], and is intimately tied to heterotic anomaly-cancellation. We return to this point in Section 4.

Beyond tree-level, the partition function receives contributions from the one-loop determinants of the kinetic operators for the fluctuating modes on C . Because of the twisting of the right-moving worldvolume fermions, the determinants associated to these fermions cancel the corresponding determinants associated to the non-zero, right-moving modes of the bosons. Equivalently, this cancellation is a consequence of the two residual supercharges preserved by C . So the non-trivial determinantal factors appearing in W_C arise only from the left-moving sector of the worldvolume theory.

In the left-moving sector, the path integral over the $SO(32)$ current algebra fermions is represented by the Pfaffian factor in the numerator of W_C . The path integral over the non-zero, left-moving modes of the worldvolume bosons is similarly represented by the product of determinants in the denominator of W_C . In these expressions, $\bar{\partial}_{V_-}$, $\bar{\partial}_{\mathcal{O}}$, and $\bar{\partial}_{\mathcal{O}(-1)}$ denote the respective $\bar{\partial}$ operators on C coupled to the corresponding holomorphic bundles. Because the bosons x^μ valued in the trivial bundle \mathcal{O}^2 have zero-modes, we include a “prime” on the determinant of $\bar{\partial}_{\mathcal{O}}$ to indicate that this determinant is to be computed with the zero-mode omitted; it otherwise vanishes. The determinants of right-moving fields cancel between bosons and fermions, which is why (3.7) contains the determinants only of $\bar{\partial}$ operators, not ∂ operators.

Finally, to extend the formula for W_C in (3.7) to the $E_8 \times E_8$ heterotic string, we simply reinterpret the Pfaffian factor as the partition function of the $\text{Spin}(32)/\mathbb{Z}_2$ current

algebra at level one, coupled to the background gauge field. Then the $E_8 \times E_8$ analogue of this factor is the partition function of the $E_8 \times E_8$ current algebra at level one, coupled to the background gauge field.

Vanishing of W_C

The $\bar{\partial}$ operators that appear in W_C depend holomorphically on the complex structure moduli of X and the bundle V . Consequently, the corresponding one-loop determinants in W_C depend holomorphically on these moduli, though we have not made this explicit in the notation. In particular examples, with some work this dependence can be explicitly evaluated. See [2,3] for some elegant computations of the Pfaffian factor in the case that X is elliptically fibered.

However, with no work at all we can deduce an important result from our formal expression for W_C — namely, the condition for W_C to vanish. Since in the present paper we are fundamentally concerned with showing that the higher F -term analogues of W_C are non-zero, let us recall under what conditions W_C is zero or non-zero.

From the formula in (3.7), we see immediately that W_C vanishes if and only if the operator $\bar{\partial}_{V_-}$ has at least one zero-mode, since the Pfaffian factor vanishes in that case. In turn, as originally noted in work of Distler and Greene [29], this observation implies that W_C vanishes if and only if the restriction of V to C is non-trivial.

To elucidate the latter point following [12], we recall that any holomorphic vector bundle over $C = \mathbb{CP}^1$ splits as a direct sum of line-bundles. Thus the restriction of the $SO(32)$ bundle V to C takes the form

$$V|_C = \bigoplus_{i=1}^{16} [\mathcal{O}(m_i) \oplus \mathcal{O}(-m_i)], \quad (3.8)$$

where the m_i are non-negative integers determined by V up to permutation. Without loss, we identify the left-moving spin bundle S_- on C as $S_- = \mathcal{O}(-1)$. Thus, the bundle $V_- = S_- \otimes V|_C$ takes the form

$$V_- = \bigoplus_{i=1}^{16} [\mathcal{O}(m_i - 1) \oplus \mathcal{O}(-m_i - 1)]. \quad (3.9)$$

Now, the $\bar{\partial}$ operator coupled to the line-bundle $\mathcal{O}(m)$ on \mathbb{CP}^1 has $m + 1$ zero-modes for $m \geq 0$ and no zero-modes otherwise. So from (3.9) we find that the dimension of the kernel of the operator $\bar{\partial}_{V_-}$ is given by

$$\dim_{\mathbb{C}} \text{Ker}(\bar{\partial}_{V_-}) = \sum_{i=1}^{16} m_i. \quad (3.10)$$

This kernel vanishes if and only if all the m_i vanish, in which case $V|_C$ is trivial.

3.3. Computing the Metric on Hypermultiplet Moduli Space

We now consider a computation with $\mathcal{N} = 2$ supersymmetry but which, as we will explain in Section 4, is a good starting point for understanding how to generate multi-fermion F -terms from worldsheet instantons.

We take our Calabi-Yau threefold to be the product $X = E \times Y$ of an elliptic curve E and a $K3$ surface Y . To ensure supersymmetry, we further assume that the gauge bundle V over X respects the product structure of X , so that V also factorizes as the tensor product of a flat bundle V_E over E and a holomorphic bundle V_Y over Y (or more generally as a direct sum of such tensor products). Heterotic compactification on X preserves $\mathcal{N} = 2$ supersymmetry in four dimensions, so worldsheet instantons will not generate a superpotential. Instead, they generate a correction to the metric on the hypermultiplet moduli space. We denote the hypermultiplet moduli space by \mathcal{M}_H .

As reviewed by Aspinwall [30], the massless hypermultiplets that arise from compactification on X describe the moduli associated to the $K3$ surface Y and its bundle V_Y , while the moduli associated to the elliptic curve E , its bundle V_E , and the dilaton itself transform in vector multiplets. Quantum corrections to the metric on hypermultiplet moduli space cannot depend on the vector multiplets, so we deduce that these corrections cannot depend on the volume of E nor on the string coupling. Because of the first fact, we can decompactify E and consider heterotic compactification to \mathbb{C}^3 on Y alone to compute the metric on \mathcal{M}_H . Because of the second fact, any corrections to the metric on \mathcal{M}_H must arise at tree-level in the string genus expansion. Combining these facts, a worldsheet instanton which can correct the metric on \mathcal{M}_H must arise from a genus zero, supersymmetric surface C in Y .

As a hyper-Kahler manifold, Y has a family of complex structures parameterized by \mathbb{CP}^1 . Any Riemann surface $C \subset Y$ is holomorphic at most in one of these complex structures. A worldsheet instanton that wraps on C preserves four supercharges if C is holomorphic in one of the complex structures, and otherwise preserves none. Which supersymmetries are preserved depends on the complex structure in which C is holomorphic.

As explained in Section 3 of [12], genus zero curves C that are holomorphic in some complex structure have a purely topological characterization. They are in one-to-one correspondence with classes l in $H_2(Y; \mathbb{Z})$, a lattice of signature $(3, 19)$, which satisfy $l^2 = -2$.

We now compute the chiral correction to the low-energy effective action generated by a worldsheet instanton wrapping such a curve C . This computation is more or less

identical in form to our previous computation of the superpotential, and we proceed in complete analogy to that case.

First, from the viewpoint of compactification on $X = E \times Y$, the normal bundle to C in X is now $N = \mathcal{O} \oplus \mathcal{O}(-2)$, where the trivial factor \mathcal{O} in N arises from the holomorphic tangent direction to E . Consequently C now carries three complex bosonic zero-modes and four right-moving fermionic zero-modes. The extra bosonic and fermionic zero-modes arise from the boson y^m and the fermions $\bar{\chi}_{\dot{\alpha}}^m$ tangent to E .

As before, these zero-modes arise as Goldstone modes for the six broken translation symmetries along $\mathbb{C}^2 \times E$ and for the four broken supersymmetries. In the case of heterotic compactification to \mathbb{C}^3 on Y itself, these zero-modes generate the usual six-dimensional chiral measure $d^6x d^4\tilde{\theta}$. We write $\tilde{\theta}$ for the six-dimensional chiral spinor to distinguish it from the four-dimensional chiral spinor θ that has already appeared. For simplicity, we write the formula below in this six-dimensional notation, without distinguishing whether the d^6x integral is over $\mathbb{C}^2 \times E$ or \mathbb{C}^3 .

We compute the chiral correction to the low-energy effective action again by evaluating the one-loop partition function of the worldvolume theory on C . In complete analogy to (3.6) and (3.7), we find

$$\delta S = \int d^6x d^4\tilde{\theta} \Psi_C, \quad (3.11)$$

where

$$\Psi_C = \exp \left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B \right) \frac{\text{Pfaff}(\bar{\partial}_{V_-})}{(\det' \bar{\partial}_{\mathcal{O}})^3 (\det' \bar{\partial}_{\mathcal{O}(-2)})}. \quad (3.12)$$

Here $V_- \equiv S_- \otimes V_Y|_C$.

The only way in which the formula for Ψ_C in (3.12) differs from the formula for W_C in (3.7) is through the determinants appearing in the denominators of these expressions. In the case at hand, the normal bundle to C in ten dimensions is $\mathcal{O}^3 \oplus \mathcal{O}(-2)$, and the left-moving modes of the three complex bosons valued in the rank three trivial bundle \mathcal{O}^3 contribute the factor $(\det' \bar{\partial}_{\mathcal{O}})^3$ in (3.12). Again, the “prime” indicates that we omit the zero-mode when computing this determinant.

The kinetic operator for the fourth boson, which describes fluctuations normal to C inside Y , is the Laplacian on C coupled to the line-bundle $\mathcal{O}(-2)$. We factor this Laplacian as $\Delta_{\mathcal{O}(-2)} = \partial_{\mathcal{O}(-2)} \bar{\partial}_{\mathcal{O}(-2)}$. Although the Laplacian $\Delta_{\mathcal{O}(-2)}$ itself has no kernel or cokernel, its factors do, so that we must write $\det \Delta_{\mathcal{O}(-2)} = \det' \partial_{\mathcal{O}(-2)} \det' \bar{\partial}_{\mathcal{O}(-2)}$. Again, the unbroken supersymmetries on the worldvolume imply that the right-moving

fermions cancel the factor $\det' \partial_{\mathcal{O}(-2)}$, leaving the factor $\det' \bar{\partial}_{\mathcal{O}(-2)}$ in the denominator of Ψ_C .

Vanishing of Ψ_C

Because W_C and Ψ_C have the same form, our discussion of the vanishing condition for W_C immediately applies to Ψ_C as well. Thus, Ψ_C vanishes if and only if the restriction of the bundle V_Y to C is nontrivial. As a positive corollary, worldsheet instantons *can* correct the metric on \mathcal{M}_H when $V_Y|_C$ is trivial. For an explicit example of how this correction to the metric on \mathcal{M}_H can be determined when S is an A_1 ALE space and V_Y is trivial, see [31].

We revisit this discussion of the instanton correction to the metric on \mathcal{M}_H in much more detail in the next section.

4. A Quantum Deformation From a Family of Rational Curves

We now discuss a new worldsheet instanton effect in heterotic Calabi-Yau compactifications. As promised, this instanton effect is a quantum deformation of the moduli space of the compactification, and it will be generated by a one-parameter family of rational curves in the threefold X . Our fundamental goal in this section is to compute the F -term correction to the effective action which is generated by such a family.

In fact, we have already performed in Section 3 a calculation that can serve as a prototype, namely the calculation that involved heterotic string compactification with $\mathcal{N} = 2$ supersymmetry on a $K3$ surface Y . From the perspective of the product threefold $X = E \times Y$, an isolated rational curve C in Y is associated to a one-parameter holomorphic family of rational curves in X , parametrized by the elliptic curve E . In this case, the trivial family $\mathcal{F} = C \times E$ generates the correction Ψ_C in (3.12) to the metric on hypermultiplet moduli space.

In the general case, we consider an arbitrary threefold X which contains an embedded rational surface \mathcal{F} , the total space of the family. By definition, \mathcal{F} can be presented as a fibration

$$\begin{array}{ccc} C & \rightarrow & \mathcal{F} \\ & & \downarrow , \\ & & \mathcal{B} \end{array} \quad (4.1)$$

where C is a rational curve in the family, and where the base \mathcal{B} is also a holomorphic curve. When we draw such a diagram to schematically represent a fibration, we understand that \mathcal{B} is the base of the fibration, \mathcal{F} is the total space, and C represents a generic fiber.

In fact, if the family is to persist under a generic complex structure deformation of X , then the base \mathcal{B} must also be rational. Otherwise, as shown by Wilson [32] and discussed in a physical context in [33], if \mathcal{B} has genus $g \geq 1$ then the family breaks up into a collection of $(2g - 2)$ isolated rational curves under a generic deformation of X . Because we are interested in families of curves which are generically present regardless of the particular complex structure on X , we assume that \mathcal{B} is also a rational curve.

The instanton calculation does not depend on all of the details of the Calabi-Yau threefold X , but only on the behavior near the surface \mathcal{F} . In order to simplify the instanton computation, we consider only local, non-compact models for X . A small neighborhood of \mathcal{F} in X can be represented as a complex line bundle over \mathcal{F} or in other words as a fiber bundle in which the generic fiber is a copy of \mathbb{C} :

$$\begin{array}{ccc} \mathbb{C} & \rightarrow & X \\ & & \downarrow \\ & & \mathcal{F}. \end{array} \quad (4.2)$$

As simple examples, we might take \mathcal{F} to be a Hirzebruch or a del Pezzo surface, for which our local model of X can actually be embedded globally in a compact, elliptically-fibered threefold. See [34,35] for some concrete examples of such threefolds (applied in these references in the context of F -theory).

The assumption that X is non-compact accomplishes two things. First, we decouple gravity in four dimensions, and our analysis in Section 2 of multi-fermion F -terms in globally supersymmetric theories is applicable. Second, as \mathcal{F} itself is a fibration over the rational curve \mathcal{B} , the threefold X has additionally the structure of a local $K3$ fibration over \mathcal{B} ,

$$\begin{array}{ccc} Y & \rightarrow & X \\ & & \downarrow \\ & & \mathcal{B}, \end{array} \quad (4.3)$$

where the generic fiber Y is an A_1 ALE space containing the rational curve C . In this situation, a simple and economical way to perform the instanton computation on X is to apply fiberwise our previous instanton computation on the $K3$ surface Y .

4.1. More About the Instanton Correction to the Metric on Hypermultiplet Moduli Space

In Section 3.3, we expressed the instanton correction to the metric on the hypermultiplet moduli space \mathcal{M}_H in the manifestly $\mathcal{N} = 2$ supersymmetric form below,

$$\delta S = \int d^6x d^4\tilde{\theta} \Psi_C. \quad (4.4)$$

In order to apply this result fiberwise to the case of a family in X , we now reinterpret the formula (4.4) in a way which generalizes to a situation with only $\mathcal{N} = 1$ supersymmetry in four dimensions.

Reducing From $\mathcal{N} = 2$ to $\mathcal{N} = 1$

Let us start by rewriting δS in the notation of Section 3.1, which is more appropriate for compactification to four dimensions $X = E \times Y$. Thus,

$$\delta S = \int d^4x d^2y d^2\theta d^2\bar{\chi} \Psi_C. \quad (4.5)$$

Here d^2y is the measure on the elliptic curve E induced from the background metric on E , and $d^2\theta d^2\bar{\chi}$ is the reduction to four dimensions of the six-dimensional chiral measure $d^4\tilde{\theta}$. In order to write δS in a form appropriate for a four-dimensional action with only $\mathcal{N} = 1$ supersymmetry, we obviously need to perform the bosonic integral over E and the fermionic integral with respect to $d^2\bar{\chi}$.

The bosonic integral over E in (4.5) is trivial, since nothing in the integrand depends on E . This integral produces a factor of the area of E , which is then reabsorbed when we rescale the four-dimensional metric to Einstein frame. As we have already observed, any correction to the metric on \mathcal{M}_H cannot depend on the vector multiplets which describe the moduli associated to E .

In contrast to the bosonic integral over E , the fermionic integral with respect to $d^2\bar{\chi}$ is quite interesting. From the perspective of a perturbative worldvolume computation, the latter integral is the integral over the zero-modes of the worldvolume fermions $\bar{\chi}_{\dot{\alpha}}^m$ tangent to E , so performing this integral implicitly reveals how the fermionic zero-modes associated to the family are “soaked up” in worldvolume perturbation theory. Of course, at this point we could perform such an analysis directly by considering the various interaction terms involving $\bar{\chi}_{\dot{\alpha}}^m$ in the worldvolume Green-Schwarz action. However, a much more elegant approach is to use the structure of $\mathcal{N} = 2$ supersymmetry present in this example.

To explain this approach, let us begin by asking a naive question. Implicit in our expression for δS in (4.5) is the choice of a distinguished $\mathcal{N} = 1$ subalgebra of the full $\mathcal{N} = 2$ supersymmetry algebra. This subalgebra is the subalgebra associated to the superspace coordinates θ^α and their conjugates, and it is the subalgebra that we have chosen to keep manifest when we reduce from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. So we ask — how does this distinguished subalgebra arise?

We recall from Section 2 that the effective action describing motion on the hypermultiplet moduli space \mathcal{M}_H is a four-dimensional nonlinear sigma model describing maps $\Phi : \mathbb{R}^4 \longrightarrow \mathcal{M}_H$. Because \mathcal{M}_H is a hyperkahler manifold, this sigma model preserves $\mathcal{N} = 2$ supersymmetry. The fact that \mathcal{M}_H is hyperkahler deserves comment, since in $\mathcal{N} = 2$ supergravity \mathcal{M}_H is only quaternionic Kahler [36]. Here we use the fact that we work with a local, non-compact model for Y (and also X) so that gravity is effectively decoupled.

Although a general quaternionic Kahler manifold need not be Kahler, a hyperkahler manifold certainly is Kahler. Our choice of a distinguished $\mathcal{N} = 1$ subalgebra in (4.5) then corresponds geometrically to the choice of a distinguished complex structure on \mathcal{M}_H , in which we regard \mathcal{M}_H as an ordinary Kahler manifold appropriate for a sigma model with only $\mathcal{N} = 1$ supersymmetry.

The distinguished complex structure on \mathcal{M}_H arises in turn from a distinguished complex structure on Y — namely, the complex structure on Y in which the surface C generating Ψ_C in (4.5) is holomorphic. For later reference, we find it useful to make the complex structure induced on \mathcal{M}_H from $C \subset Y$ explicit. For this, it suffices to describe the local holomorphic tangent space at each point in \mathcal{M}_H .

First, we pick a basepoint in \mathcal{M}_H associated to the distinguished complex structure on Y . At this point, the holomorphic tangent space to \mathcal{M}_H is spanned by elements of the following Dolbeault cohomology groups, all of which are defined using the given complex structure on Y ,

$$\delta A \in H_{\bar{\partial}}^1(Y, \text{End } V_Y), \quad \delta T \in H_{\bar{\partial}}^1(Y, \Omega_Y^1), \quad \delta U \in H_{\bar{\partial}}^1(Y, TY). \quad (4.6)$$

Here δA describes an infinitesimal deformation of the vector bundle V_Y , δT describes an infinitesimal change in the complexified Kahler class of Y , and δU describes an infinitesimal deformation of the complex structure of Y .

Having described the holomorphic tangent space at the basepoint in \mathcal{M}_H via (4.6), we now use the background hyperkahler metric on \mathcal{M}_H to transport each of the holomorphic tangent vectors δA , δT , and δU to every other point in \mathcal{M}_H , where their span defines the holomorphic tangent space at these other points. This definition requires a choice of paths connecting the basepoint to every other point in \mathcal{M}_H for parallel transport, but since the metric on \mathcal{M}_H is hyperkahler, the resulting span of δA , δT , and δU at each point is independent of the path.

Performing the Fermionic Integral

Because the fermionic measure $d^2\bar{\chi}$ can be interpreted as part of the chiral measure on $\mathcal{N} = 2$ superspace, the corresponding fermionic integral can be equivalently evaluated by acting on the integrand Ψ_C with the operator $\{\overline{Q}_{\dot{\alpha}}^{(2)}, [\overline{Q}^{(2)\dot{\alpha}}, \cdot]\}$, where $\overline{Q}_{\dot{\alpha}}^{(2)}$ is the anti-chiral supercharge generating translations along $\bar{\chi}$ in superspace. We distinguish this anti-chiral supercharge from the anti-chiral supercharge $\overline{Q}_{\dot{\alpha}}^{(1)} \equiv \overline{Q}_{\dot{\alpha}}$ which is part of the distinguished $\mathcal{N} = 1$ subalgebra.

To describe geometrically the action of $\overline{Q}_{\dot{\alpha}}^{(2)}$ in the hyperkahler sigma model, we again introduce $\mathcal{N} = 1$ chiral and anti-chiral superfields Φ^i and $\overline{\Phi}^{\bar{i}}$ to describe local holomorphic and anti-holomorphic coordinates on \mathcal{M}_H in the distinguished complex structure. This complex structure corresponds to a covariantly constant endomorphism \mathbf{I} of $T\mathcal{M}_H$ satisfying $\mathbf{I}^2 = -1$, and as we saw in Section 2, the action of the associated supercharge $\overline{Q}_{\dot{\alpha}}^{(1)}$ is identified with the action of the $\bar{\partial}$ operator on \mathcal{M}_H .

Because \mathcal{M}_H is hyperkahler, we also have covariantly constant tensors \mathbf{J} and \mathbf{K} which define additional complex structures on \mathcal{M}_H and which satisfy the quaternion algebra with \mathbf{I} . Of course, the tensor \mathbf{J} is used to define the extra supercharges of the $\mathcal{N} = 2$ supersymmetry algebra, and under the action of the supercharge $\overline{Q}_{\dot{\alpha}}^{(2)}$, the component fields ϕ^i and ψ_{β}^i of Φ^i and the component fields $\overline{\phi}^{\bar{i}}$ and $\overline{\psi}_{\dot{\beta}}^{\bar{i}}$ of $\overline{\Phi}^{\bar{i}}$ transform as

$$\begin{aligned}\delta_{\dot{\alpha}}^{(2)}\phi^i &= \mathbf{J}_{\dot{\alpha}}^i \overline{\psi}_{\dot{\alpha}}^{\bar{i}}, & \delta_{\dot{\alpha}}^{(2)}\overline{\phi}^{\bar{i}} &= 0, \\ \delta_{\dot{\alpha}}^{(2)}\psi_{\beta}^i &= i \mathbf{J}_{\dot{\alpha}}^i \partial_{\dot{\alpha}\beta} \overline{\phi}^{\bar{i}} - \Gamma_{j\bar{k}}^i \mathbf{J}_{\bar{k}}^j \psi_{\beta}^j \overline{\psi}_{\dot{\alpha}}^{\bar{j}}, & \delta_{\dot{\alpha}}^{(2)}\overline{\psi}_{\dot{\beta}}^{\bar{i}} &= 0.\end{aligned}\tag{4.7}$$

Here Γ is the connection associated to the hyperkahler metric on \mathcal{M}_H . Just as in Section 2, these supersymmetry transformations imply that the action of $\overline{Q}_{\dot{\alpha}}^{(2)}$ can be identified geometrically with the action of the differential operator $\mathbf{J}(\partial)$, or in components $\mathbf{J}_{\dot{\alpha}}^i \partial_i$, on \mathcal{M}_H .

With this geometric description of $\overline{Q}_{\dot{\alpha}}^{(2)}$, we immediately deduce that

$$\int d^2\bar{\chi} \Psi_C = \left\{ \overline{Q}_{\dot{\alpha}}^{(2)}, [\overline{Q}^{(2)\dot{\alpha}}, \Psi_C] \right\} = \mathbf{J}_{\dot{\alpha}}^i \mathbf{J}_{\bar{j}}^j (\nabla_i \nabla_j \Psi_C) \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{i}} \overline{D}^{\dot{\alpha}} \overline{\Phi}^{\bar{j}}. \tag{4.8}$$

Here ∇ denotes the covariant derivative associated to the background hyperkahler metric on \mathcal{M}_H . In writing (4.8), we note that Ψ_C transforms globally as a function on \mathcal{M}_H , and we use the fact that \mathbf{J} is covariantly constant and so annihilated by ∇ .

In the notation of a four-dimensional effective action with only $\mathcal{N} = 1$ supersymmetry, the instanton correction δS is then given by

$$\delta S = \int d^4x d^2\theta \mathbf{J}_{\dot{\alpha}}^i \mathbf{J}_{\bar{j}}^j (\nabla_i \nabla_j \Psi_C) \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{i}} \overline{D}^{\dot{\alpha}} \overline{\Phi}^{\bar{j}}. \tag{4.9}$$

Hence the trivial family of instantons parametrized by E generates the multi-fermion F -term of degree $p = 1$ that is represented geometrically by the tensor

$$\omega_{\bar{i}\bar{j}} = \mathbf{J}_i^{\bar{i}} \mathbf{J}_{\bar{j}}^j (\nabla_i \nabla_j \Psi_C) . \quad (4.10)$$

A Geometric Example

The expression for ω in (4.10) may appear somewhat abstract, so let us present a simple geometric example in which this formula has a clear interpretation.

We first require a hyperkahler model for \mathcal{M}_H . For this model, we introduce an arbitrary compact Kahler manifold M , with Kahler metric $ds^2 = h_{k\bar{k}} dz^k d\bar{z}^{\bar{k}}$, where z^k and $\bar{z}^{\bar{k}}$ are local holomorphic and anti-holomorphic coordinates on M . We then model \mathcal{M}_H on a formal neighborhood of M embedded as the zero-section in the total space of its holomorphic cotangent bundle T^*M .

The holomorphic cotangent bundle is canonically a holomorphic symplectic manifold, meaning that it admits a canonical holomorphic symplectic form Ω . To describe Ω explicitly, we introduce local holomorphic and anti-holomorphic coordinates p_k and $\bar{p}_{\bar{k}}$ on the fibers of T^*M . Then $\Omega = dz^k \wedge dp_k$, where we apply the Einstein summation convention for the index ‘ k ’. This two-form is manifestly well-defined, holomorphic, and non-degenerate on the holomorphic tangent space. Any hyperkahler manifold (regarded as a Kahler manifold in a fixed complex structure) possesses such a holomorphic symplectic form, and as proven recently in [37,38] and discussed physically in [39,40], a formal neighborhood of M in T^*M admits a hyperkahler metric whose restriction to M coincides with the given Kahler metric h and whose holomorphic symplectic form coincides with the canonical form Ω .

One reason to consider this cotangent model for \mathcal{M}_H (besides its simplicity) is that this model actually arises in practice as a local model for an open set in the moduli space of bundles on a $K3$ surface. See Section 2.4 of [41] for an explanation of this fact.

We are interested in corrections to the hyperkahler metric on \mathcal{M}_H , and to describe these corrections in the cotangent model, we naturally consider an infinitesimal deformation of the complex structure of the base M . As in Section 2, this deformation is represented by a tensor $\omega_{\bar{l}}^k d\bar{z}^{\bar{l}} \otimes \partial/\partial z^k$ on M , for which the induced correction to the background Kahler metric h on M is

$$\delta h_{\bar{k}\bar{l}} = \omega_{\bar{k}\bar{l}} = \frac{1}{2} (h_{k\bar{k}} \omega_{\bar{l}}^k + h_{\bar{k}\bar{l}} \omega_{\bar{k}}^k) . \quad (4.11)$$

(There is also conjugate correction δh_{kl} which we suppress.) Since the deformation of M is represented by a correction to the Kahler metric on M , the induced hyperkahler metric on T^*M is similarly corrected.

Our goal is now to explain how this deformation of the complex structure on M is related to a function Ψ_C as in (4.10). This function must be determined by the tensor ω , and at least to leading order near $M \subset T^*M$, we can make a natural guess for Ψ_C ,

$$\Psi_C = \omega_{\bar{l}}^k h^{\bar{l}l} p_k p_l + \dots \quad (4.12)$$

Here the ‘ \dots ’ indicate higher-order terms in Ψ_C that determine the hyperkahler metric on T^*M away from M .

To check our guess for Ψ_C , we need only check that the formula (4.10) reproduces the correction $\omega_{\bar{k}\bar{l}}$ to the metric on M . For this check, we must know the tensor \mathbf{J} on T^*M as restricted to M . In general, \mathbf{J} is determined by the holomorphic symplectic form Ω and the hyperkahler metric \mathbf{g} on \mathcal{M}_H via $\mathbf{J}_i^j = (\bar{\Omega})_{\bar{i}\bar{j}} \mathbf{g}^{\bar{j}i}$. Applied to our cotangent model, for which we know Ω and the hyperkahler metric \mathbf{g} as restricted to M , we see that

$$\mathbf{J} = h^{\bar{k}k} d\bar{p}_{\bar{k}} \otimes \frac{\partial}{\partial z^k} + h_{\bar{k}k} d\bar{z}^{\bar{k}} \otimes \frac{\partial}{\partial p_k} - h^{\bar{k}k} dp_k \otimes \frac{\partial}{\partial \bar{z}^{\bar{k}}} - h_{\bar{k}k} dz^k \otimes \frac{\partial}{\partial \bar{p}_{\bar{k}}} + \dots \quad (4.13)$$

Here the ‘ \dots ’ again indicate higher-order terms that vanish when we restrict \mathbf{J} to M .

In particular, this formula for \mathbf{J} implies that $\mathbf{J}(dp_k) = h_{\bar{k}k} d\bar{z}^{\bar{k}}$ on M . Our general formula $\omega_{\bar{i}\bar{j}} = \mathbf{J}_i^{\bar{i}} \mathbf{J}_{\bar{j}}^j (\nabla_i \nabla_j \Psi_C)$ now immediately reproduces the given expression for ω in (4.11). We simply note that since Ψ_C is quadratic in the fiber coordinates p_k , the only terms in $\nabla_i \nabla_j \Psi_C$ which contribute when we restrict to M are those involving two derivatives with respect to these coordinates.

Relation To Worldvolume Perturbation Theory

The expression in (4.9) for δS as a multi-fermion F -term is quite compact and explicit. However, our derivation of the F -term in this form relies heavily on the presence of $\mathcal{N} = 2$ supersymmetry, which we used to perform the integral with respect to $d^2 \bar{\chi}$.

On the other hand, from the perspective of the worldvolume theory on C , this fermionic integral is an integral over the two zero-modes of the fermions $\bar{\chi}_{\dot{\alpha}}^{\bar{m}}$ tangent to the family E . So we could alternatively perform the integral using standard worldvolume perturbation theory. In this approach, we simply bring down interaction terms involving $\bar{\chi}_{\dot{\alpha}}^{\bar{m}}$ from the worldvolume action to absorb the pair of fermion zero-modes in the worldvolume

path integral. Although this perturbative method of computing the F -term is less elegant, it has the advantage of systematically generalizing to an honest heterotic background with only $\mathcal{N} = 1$ supersymmetry.

To make the perturbative structure of the F -term correction in (4.9) clear, we now evaluate more explicitly our formula for the tensor ω ,

$$\omega_{\bar{i}\bar{j}} = \mathbf{J}_{\bar{i}}^i \mathbf{J}_{\bar{j}}^j (\nabla_i \nabla_j \Psi_C) . \quad (4.14)$$

In order to evaluate (4.14), we first require a more explicit description of the tensor \mathbf{J} on \mathcal{M}_H .

Since \mathbf{J} is related to the holomorphic symplectic form Ω and the hyperkahler metric \mathbf{g} on \mathcal{M}_H via $\mathbf{J}_{\bar{i}}^i = (\bar{\Omega})_{\bar{i}\bar{j}} \mathbf{g}^{\bar{j}i}$, we start by describing Ω and \mathbf{g} . First, the hyperkahler metric \mathbf{g} on \mathcal{M}_H is induced in the usual way from the hyperkahler metric G on the $K3$ surface Y itself. As for Ω , this two-form is described on \mathcal{M}_H by

$$\Omega = \int_Y \Omega_Y \wedge \text{Tr}(\delta A \wedge \delta A) + \int_Y \Omega_Y \wedge \text{Tr}(\delta T \wedge \delta U) . \quad (4.15)$$

Here Ω_Y is the holomorphic two-form on Y , and δA , δT , and δU are harmonic representatives of the Dolbeault cohomology classes on Y specified in (4.6). These elements represent local holomorphic tangent vectors to \mathcal{M}_H , and in (4.15) they should be dually regarded as local holomorphic one-forms on \mathcal{M}_H . Also, Tr denotes a suitably normalized trace over the gauge indices of δA in the first term of (4.15) and a corresponding trace over the indices associated to Ω_Y^1 in δT and TY in δU in the second term of (4.15).

In local holomorphic and anti-holomorphic coordinates u^k and $\bar{u}^{\bar{k}}$ on Y , the expression for Ω becomes

$$\Omega = \int_Y d^4 u (\Omega_Y)_{k_1 k_2} \left[\delta A_{\bar{k}_1}^a \delta A_{\bar{k}_2 a} + \delta T_{\bar{k}_1 k_3} \delta U_{\bar{k}_2}^{k_3} \right], \quad d^4 u \equiv du^{k_1} \wedge du^{k_2} \wedge d\bar{u}^{\bar{k}_1} \wedge d\bar{u}^{\bar{k}_2} . \quad (4.16)$$

Here a denotes the adjoint-valued gauge index on A .

Given this presentation of Ω , we deduce that tensor \mathbf{J} on \mathcal{M}_H is given in local coordinates by

$$\begin{aligned} \mathbf{J} = \int_Y d^4 u (\bar{\Omega}_Y)_{\bar{k}_1 \bar{k}_2} & \left[G_{\bar{k}_3 k_2} \delta \bar{A}_{\bar{k}_1}^a \otimes \frac{\delta}{\delta A_{\bar{k}_3}^a} + G_{\bar{k}_3 k_3} G_{\bar{k}_4 k_2} \delta \bar{U}_{\bar{k}_1}^{\bar{k}_3} \otimes \frac{\delta}{\delta T_{\bar{k}_4 k_3}} + \right. \\ & \left. + G_{\bar{k}_4 k_2} G^{\bar{k}_3 k_4} \delta \bar{T}_{\bar{k}_3 k_1} \otimes \frac{\delta}{\delta U_{\bar{k}_4}^{k_4}} \right] + \dots \end{aligned} \quad (4.17)$$

For brevity, we have only written the tensor components of the form $\mathbf{J}_{\bar{i}}^i$ in (4.17), the omitted components $\mathbf{J}_{\bar{i}}^{\bar{i}}$ being determined by the relation $\mathbf{J}^2 = -1$.

We interpret the tensor \mathbf{J} as an operator acting on the partition function Ψ_C via the differentials $\delta/\delta A$, $\delta/\delta T$, and $\delta/\delta U$. Since Ψ_C is computed from the free worldvolume action I_C , the variation of Ψ_C on \mathcal{M}_H is determined by the corresponding variation of I_C , which is given explicitly by

$$\begin{aligned}\delta I_C = & \frac{1}{2\pi\alpha'} \int_C d^2z \operatorname{Tr}(\delta A_{\bar{z}} j_z) + \frac{1}{2\pi\alpha'} \int_C d^2z \delta T_{\bar{z}m} \partial_z y^m + \\ & + \frac{1}{2\pi\alpha'} \int_C d^2z \frac{1}{2} (\delta U_{\bar{z}}^k G_{\bar{m}k} + \delta U_{\bar{m}}^k G_{\bar{z}k}) \partial_z \bar{y}^{\bar{m}} + \mathcal{O}(y^2).\end{aligned}\quad (4.18)$$

Here j_z is an element of the left-moving current algebra associated to the heterotic gauge bundle, and we recall from Section 3.1 that y^m and $\bar{y}^{\bar{m}}$ are worldvolume bosons valued in the normal bundle to C in Y . Note that we use the notation z for the tangent index to C , m for an index in the normal bundle to C in Y , and k for a general tangent index to Y .

In writing δI_C , we only present the terms of first-order in the fluctuating bosons y and \bar{y} , so that all terms in δI_C have the form of a holomorphic coupling between δA , δT , or δU and a corresponding left-moving current on C . Non-holomorphic, quadratic couplings involving these bosons, such as $\int_C d^2z \delta T_{\bar{m}m} \partial_{\bar{z}} \bar{y}^{\bar{m}} \partial_z y^m$, are irrelevant in the weak coupling limit $\alpha' \rightarrow 0$ and hence do not contribute when \mathbf{J} acts on Ψ_C . Although we have made the conventional normalization by $1/2\pi\alpha'$ explicit in δI_C , we henceforth suppress this factor to avoid cluttering the following formulae.

Finally, the variation δI_C in (4.18) does not include any terms from the right-moving worldvolume fermions. The kinetic terms for these fermions involve the ∂ operator on C , and this operator varies anti-holomorphically on \mathcal{M}_H . Consequently, it is invariant under the holomorphic variations δA , δT , and δU .

From the formula for \mathbf{J} in (4.17) and the formula for δI_C in (4.18), we directly compute the action of \mathbf{J} on the functional I_C to be

$$\begin{aligned}\mathbf{J}(I_C) = & \int_C d^2z (\bar{\Omega}_Y)_{\bar{k}\bar{z}} G^{\bar{k}k} \operatorname{Tr}(\delta \bar{A}_k j_z) + \int_C d^2z (\bar{\Omega}_Y)_{\bar{k}_1 \bar{z}} G^{\bar{k}_1 k_1} G_{\bar{k}_2 m} \delta \bar{U}_{k_1}^{\bar{k}_2} \partial_z y^m + \\ & + \int_C d^2z \frac{1}{2} [(\bar{\Omega}_Y)_{\bar{k}\bar{z}} G^{\bar{k}k} \delta \bar{T}_{\bar{m}k} + (\bar{\Omega}_Y)_{\bar{k}\bar{m}} G^{\bar{k}k} \delta \bar{T}_{\bar{z}k}] \partial_z \bar{y}^{\bar{m}} + \mathcal{O}(y^2).\end{aligned}\quad (4.19)$$

When \mathbf{J} acts on Ψ_C , the terms in $\mathbf{J}(I_C)$ above are pulled down as insertions into the worldvolume path integral. Hence in perturbation theory, these terms must arise from

interaction terms I_C^{int} in the Green-Schwarz worldvolume action which involve the fermions $\bar{\chi}_{\dot{\alpha}}^{\overline{m}}$ with zero-modes. We will explain this point of view further when we consider the generalization of this computation to $\mathcal{N} = 1$ backgrounds.

Thus, to evaluate the tensor ω in perturbation theory, we compute the worldvolume path integral on C with two insertions of the source terms in $\mathbf{J}(I_C)$. This computation amounts to the evaluation (as a function of the moduli) of the following left-moving current-current correlators on C ,

$$\begin{aligned} \omega = & \int_{C \times C} d^2z d^2w \left[(\bar{\Omega}_Y)_{\bar{k}_1 \bar{z}} G^{\bar{k}_1 k_1} \delta \bar{A}_{k_1}^a \right] \left[(\bar{\Omega}_Y)_{\bar{k}_2 \bar{w}} G^{\bar{k}_2 k_2} \delta \bar{A}_{k_2}^b \right] \langle j_{za} j_{wb} \rangle' + \\ & + \int_{C \times C} d^2z d^2w \left[(\bar{\Omega}_Y)_{\bar{k}_1 \bar{z}} G^{\bar{k}_1 k_1} G_{\bar{k}_3 m} \delta \bar{U}_{k_1}^{\bar{k}_3} \right] \left[\frac{1}{2} (\bar{\Omega}_Y)_{\bar{k}_2 \bar{w}} G^{\bar{k}_2 k_2} \delta \bar{T}_{\bar{m} k_2} + (\bar{w} \leftrightarrow \bar{m}) \right] \times \\ & \times \langle \partial_z y^m \partial_w \bar{y}^{\bar{m}} \rangle'. \end{aligned} \tag{4.20}$$

Here $\langle \dots \rangle'$ indicates the expectation value as evaluated in the free theory on C , with bosonic zero-modes omitted. Indeed, since these current-current correlators are evaluated in the free theory, we omit the obvious cross-terms from $\mathbf{J}(I_C)^2$ which involve such quantities as $\langle j_{za} \partial_w y^m \rangle'$ and thus vanish identically.

At this point, one might ask what we can hope to learn from such an unwieldy expression as (4.20). First, this quantity is manifestly non-zero in general, though even this fact is much clearer from the simple expression in (4.14).

Second, we see that the tensor ω on \mathcal{M}_H has an interesting structure. From the perspective of the distinguished complex structure on \mathcal{M}_H , the second term in (4.20) indicates that ω has components with “mixed” indices involving both the complex structure and complexified Kahler moduli of Y . So locally on \mathcal{M}_H , the deformation mixes these moduli. Globally, since \mathcal{M}_H is hyperkahler, there is no real distinction anyway between these two local sets of moduli. Nonetheless, we will find the same result when we consider next the $\mathcal{N} = 1$ supersymmetric generalization of this computation. In that case, this geometric fact about the structure of the deformation has a global significance.

4.2. Generalization to $\mathcal{N} = 1$ Backgrounds

We now extend our discussion to the general case that X is a local Calabi-Yau threefold which contains an embedded one-parameter family \mathcal{F} of rational curves. We recall that \mathcal{F} is a fibration,

$$\begin{array}{ccc} C & \rightarrow & \mathcal{F} \\ & & \downarrow \\ & & \mathcal{B}, \end{array} \tag{4.21}$$

where C is a rational curve in the family, and we assume that the base curve \mathcal{B} is also rational. The normal bundle to the generic curve C in X is again $N = \mathcal{O} \oplus \mathcal{O}(-2)$, and the structure of the right-moving fermion zero-modes on C is the same as in the case $X = E \times Y$.

Since worldsheet instantons in the background with $\mathcal{N} = 2$ supersymmetry can generate a correction which reduces to the relevant multi-fermion F -term, we certainly expect that instantons in a background with only $\mathcal{N} = 1$ supersymmetry can also generate the F -term. Our goal is to make this intuition precise by explaining how the F -term can generally be computed. We start very naively with a local analysis in worldvolume perturbation theory on a given curve C in the family \mathcal{F} . We then extend this local analysis to provide a more global, geometric description of the tensor ω , analogous to our result that $\omega_{\overline{i}\overline{j}} = \mathbf{J}_{\overline{i}}^i \mathbf{J}_{\overline{j}}^j (\nabla_i \nabla_j \Psi_C)$ in the example with $\mathcal{N} = 2$ supersymmetry.

Local Analysis in Worldvolume Perturbation Theory

The basic issue in a perturbative instanton computation on C is to understand how the two zero-modes of the worldvolume fermions $\overline{\chi}_{\dot{\alpha}}^m$ tangent to the base \mathcal{B} of the family can be soaked up from interaction terms in the worldvolume action. Of course to answer this question, we must know something about the higher-order terms in the full physical-gauge Green-Schwarz action on C in an arbitrary (on-shell) supergravity background. In the case of the heterotic string, these couplings were systematically computed some time ago by Grisaru and collaborators [42,43] using superspace techniques, and these ideas have since been extended to the worldvolume actions of Type II strings and branes. For a short recent review of this subject, see [44] and references therein.

In the case at hand, worldvolume couplings of the proper form to generate the F -term are quite limited. For instance, we expect via holomorphy that these interactions couple the fermion $\overline{\chi}_{\dot{\alpha}}^m$ to a left-moving current on C . Also, since the F -term is described by a tensor $\omega_{\overline{i}\overline{j}}$, the interactions must involve the local one-forms $\delta\overline{A}$, $\delta\overline{T}$, and $\delta\overline{U}$. Our notation is as in Section 4.1, so that δA , δT , and δU describe moduli associated respectively to the bundle V , to the complexified Kahler class of X , and to the complex structure of X . Finally, because the local geometry of C in X is identical to the local geometry of C in the product $E \times Y$, whatever worldvolume couplings we consider must reduce directly to the terms in (4.19) which we deduced as a consequence of $\mathcal{N} = 2$ supersymmetry.

Because of these restrictions, the most economical way to deduce the required couplings is simply to make the natural guess, which can then be verified directly using the results in [42,43]. These couplings are given by

$$\begin{aligned} I_C^{\text{int}} = & \int_C d^2z (\bar{\Omega}_X)_{\bar{k}\bar{n}\bar{z}} G^{\bar{k}k} \left(\bar{\chi}_{\dot{\alpha}}^{\bar{n}} \bar{D}^{\dot{\alpha}} \bar{\Phi}_A^{(\bar{i}_1)} \right) \text{Tr} \left(\delta \bar{A}_{k(\bar{i}_1)} j_z \right) + \\ & + \int_C d^2z (\bar{\Omega}_X)_{\bar{k}_1 \bar{n} \bar{z}} G^{\bar{k}_1 k_1} G_{\bar{k}_2 m} \left(\bar{\chi}_{\dot{\alpha}}^{\bar{n}} \bar{D}^{\dot{\alpha}} \bar{\Phi}_U^{(\bar{i}_2)} \right) \delta \bar{U}_{k_1(\bar{i}_2)}^{\bar{k}_2} \partial_z y^m + \\ & + \int_C d^2z \left[\frac{1}{2} (\bar{\Omega}_X)_{\bar{k}\bar{n}\bar{z}} G^{\bar{k}k} \left(\bar{\chi}_{\dot{\alpha}}^{\bar{n}} \bar{D}^{\dot{\alpha}} \bar{\Phi}_T^{(\bar{i}_3)} \right) \delta \bar{T}_{\bar{m}k(\bar{i}_3)} + (\bar{z} \leftrightarrow \bar{m}) \right] \partial_z \bar{y}^{\bar{m}}. \end{aligned} \quad (4.22)$$

Here Ω_X is now the holomorphic three-form on X , and as before z is a tangent index to C , m, n, \dots are normal bundle indices for C in X , and k_1, k_2, \dots are general tangent indices to X . As in Section 2, $\bar{D}^{\dot{\alpha}} \bar{\Phi}_A^{(\bar{i})}$ denotes the four-dimensional fermion associated to the particular bundle deformation $\delta \bar{A}_{(\bar{i})}$, labelled by the index \bar{i} . From the ten-dimensional perspective, this combination $\bar{D}^{\dot{\alpha}} \bar{\Phi}_A^{(\bar{i})} \otimes \delta \bar{A}_{(\bar{i})}$ arises from the reduction of the ten-dimensional gaugino on X , where we apply the standard identification of spinors with differential forms on X . The same comments apply to the four-dimensional fermions $\bar{D}^{\dot{\alpha}} \bar{\Phi}_U^{(\bar{i})}$ and $\bar{D}^{\dot{\alpha}} \bar{\Phi}_T^{(\bar{i})}$ which arise from the reduction of the ten-dimensional gravitino.

As a simple check, in the case that $X = E \times Y$, we set $\Omega_X = dt \wedge \Omega_Y$, where dt is a holomorphic one-form on E . We then replace the worldvolume fermions $\bar{\chi}_{\dot{\alpha}}^{\bar{n}}$ with their zero-modes tangent to E . Upon factoring out quantities such as $(\bar{\chi}_{\dot{\alpha}}^{\bar{n}} \bar{D}^{\dot{\alpha}} \bar{\Phi}_A^{(\bar{i})})$ which are now constant on Y , we see that the interactions in (4.22) reproduce the terms in (4.19).

As a final caveat, we emphasize that in writing (4.22) we consider only standard heterotic Calabi-Yau backgrounds, with constant dilaton, with no H -flux, and with no torsion. Otherwise, as is clear from the general results of [42,43], additional couplings are present in the worldvolume action.

To compute the F -term perturbatively on C , we use a pair of the interaction terms in (4.22) to absorb the zero-modes of $\bar{\chi}_{\dot{\alpha}}^{\bar{m}}$, and we evaluate a current-current correlator in the left-moving sector on C . As a result, the local contribution to ω from a given curve in the family is given formally by an expression generalizing what we found in (4.20),

$$\begin{aligned} \varpi(t, \bar{t}) = & \int_{C_t \times C_t} d^2z d^2w \left[(\bar{\Omega}_X)_{\bar{k}_1 \bar{m} \bar{z}} G^{\bar{k}_1 k_1} \delta \bar{t}^{\bar{m}} \delta \bar{A}_{k_1}^a \right] \left[(\bar{\Omega}_X)_{\bar{k}_2 \bar{n} \bar{w}} G^{\bar{k}_2 k_2} \delta \bar{t}^{\bar{n}} \delta \bar{A}_{k_2}^b \right] \times \\ & \times \langle j_{za} j_{wb} \rangle' + \\ & + \int_{C_t \times C_t} d^2z d^2w \left[(\bar{\Omega}_X)_{\bar{k}_1 \bar{l} \bar{z}} G^{\bar{k}_1 k_1} G_{\bar{k}_3 m} \delta \bar{t}^{\bar{l}} \delta \bar{U}_{k_1}^{\bar{k}_3} \right] \left[\frac{1}{2} (\bar{\Omega}_X)_{\bar{k}_2 \bar{n} \bar{w}} G^{\bar{k}_2 k_2} \delta \bar{t}^{\bar{n}} \delta \bar{T}_{\bar{m} k_2} + (\bar{w} \leftrightarrow \bar{m}) \right] \times \\ & \times \langle \partial_z y^m \partial_w \bar{y}^{\bar{m}} \rangle'. \end{aligned} \quad (4.23)$$

Here t and \bar{t} are local holomorphic and anti-holomorphic coordinates on the curve \mathcal{B} which parametrizes the family, and C_t is the curve in the corresponding fiber over \mathcal{B} . Because ϖ represents only the local contribution from a given curve in the family, we distinguish this quantity from the tensor ω that arises after we integrate over \mathcal{B} , an integral we discuss below.

In the perturbative formula (4.23) for ϖ , we let δt denote a section of $H_{\bar{\partial}}^0(C_t, N)$, which represents the local holomorphic tangent space to \mathcal{B} at t . The conjugate section $\delta \bar{t}^m$ in (4.23) arises from the zero-mode of $\bar{\chi}_{\alpha}^m$, and $\delta \bar{t}$ should be dually interpreted on \mathcal{B} as a local one-form of type $(0, 1)$, analogous to $\delta \bar{A}$, $\delta \bar{T}$, and $\delta \bar{U}$.

Global Analysis

The quantity $\varpi(t, \bar{t})$ in (4.23) is only the local, perturbative contribution to the F -term from a given curve C_t in the family \mathcal{F} . However, as we have naively written (4.23), it is far from clear what sort of geometric object $\varpi(t, \bar{t})$ really is and how it can be integrated over the parameter space \mathcal{B} .

To address these global questions, we now introduce the following “universal” space \mathcal{U} . This space describes how the parameter space \mathcal{B} of the family varies as we vary the moduli associated to X and V , so that \mathcal{U} is the total space of a bundle,

$$\begin{array}{ccc} \mathcal{B} & \rightarrow & \mathcal{U} \\ & & \downarrow \pi \\ & & \mathcal{M}_0. \end{array} \quad (4.24)$$

What is the space \mathcal{M}_0 ? At first glance, we might simply take \mathcal{M}_0 to be the classical, supergravity moduli space \mathcal{M}_{cl} describing compactification on X and V . However, as explained in [12,31], the algebraic structure of \mathcal{M}_{cl} already receives perturbative corrections in α' (at string tree-level). Since we are studying non-perturbative corrections here, we let \mathcal{M}_0 be the moduli space including perturbative corrections in α' .

Indeed, before we can even discuss non-perturbative corrections to \mathcal{M}_0 , we must explain how \mathcal{M}_0 itself differs from \mathcal{M}_{cl} . This difference lies fundamentally in the exotic transformation properties of the heterotic B -field, as required for anomaly-cancellation. We recall that, in order to cancel the chiral sigma model anomaly at one-loop in α' , the heterotic B -field must transform under local Lorentz and gauge transformations as

$$B \longrightarrow B + \frac{\alpha'}{2\pi} (\text{Tr } \xi F - \text{Tr } \epsilon R), \quad (4.25)$$

where ϵ and ξ are infinitesimal parameters for the Lorentz and gauge transformations, and R and F are the standard curvature tensors. As a result of (4.25), the periods of the B -field also shift under local Lorentz and gauge transformations, with the result that the phase factor $\exp(i \int_C B)$ generally transforms in a topologically nontrivial circle bundle over the complex structure moduli spaces associated to X and V .

As an example, we return to our formula for the worldsheet instanton contribution to the superpotential,

$$W_C = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B\right) \frac{\text{Pfaff}(\bar{\partial}_{V_-})}{(\det' \bar{\partial}_{\mathcal{O}})^2 (\det \bar{\partial}_{\mathcal{O}(-1)})^2}. \quad (4.26)$$

In this expression, the chiral determinants of the various $\bar{\partial}$ operators in (4.26) transform in a generally non-trivial determinant line-bundle⁵ over the complex structure moduli space associated to X and V . In order that the phase of W_C be well-defined, $\exp(i \int_C B)$ must then transform in the circle bundle associated to the inverse of the determinant line-bundle.

The fact that $\exp(i \int_C B)$ transforms in a nontrivial circle bundle over the complex structure moduli space has an immediate implication for the algebraic structure of \mathcal{M}_0 . Because the periods of the B -field enter linearly the complexified Kahler moduli of X , we deduce that the one-loop sigma model anomaly requires each of the complexified Kahler moduli to transform in the associated \mathbb{C}^* -bundle over the complex structure moduli space of X and V . Hence this perturbative effect already spoils the classical factorization between the complexified Kahler and the complex structure moduli in \mathcal{M}_{cl} .

More About $\varpi(t, \bar{t})$

We introduce the space \mathcal{U} because $\varpi(t, \bar{t})$ in (4.23) should clearly transform as a tensor on \mathcal{U} . Indeed, the local one-forms $\delta\bar{A}$, $\delta\bar{T}$, and $\delta\bar{U}$ are sections of the pullback $\pi^*\bar{\Omega}_{\mathcal{M}_0}^1$ from the cotangent bundle of \mathcal{M}_0 , and the one-form $\delta\bar{t}$ should be interpreted as a section of the relative cotangent bundle $\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1$, which can be described by the following exact sequence on \mathcal{U} ,

$$0 \longrightarrow \pi^*\Omega_{\mathcal{M}_0}^1 \longrightarrow \Omega_{\mathcal{U}}^1 \longrightarrow \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \longrightarrow 0. \quad (4.27)$$

This bundle $\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1$ pulls back to $\bar{\Omega}_{\mathcal{B}}^1$ on each fiber in (4.24).

We now claim that $\varpi(t, \bar{t})$ transforms globally on \mathcal{U} as a symmetric section of

$$\left[\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^*\bar{\Omega}_{\mathcal{M}_0}^1 \right] \otimes \left[\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^*\bar{\Omega}_{\mathcal{M}_0}^1 \right]. \quad (4.28)$$

⁵ For a review of the definition and the properties of the determinant line-bundle, see [45].

The appearance of the bundles $\overline{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1$ and $\pi^*\overline{\Omega}_{\mathcal{M}_0}^1$ is clear, since these factors are directly associated to the appearance of $\delta\bar{t}$, $\delta\bar{A}$, $\delta\bar{T}$, and $\delta\bar{U}$ in the perturbative formula (4.23) for ϖ . However, the appearance of the holomorphic bundle $\Omega_{\mathcal{U}/\mathcal{M}_0}^1$ in (4.28) is more subtle and results from the requirement that we project out bosonic zero-modes from the expectation values $\langle \cdots \rangle'$ in (4.23).

In order to explain why ϖ transforms on \mathcal{U} as above, we find it useful to offer another, more global description of ϖ that generalizes our discussion of the example with $\mathcal{N} = 2$ supersymmetry, for which

$$\omega = (\mathbf{J} \circ \mathbf{J})(\Psi_C). \quad (4.29)$$

Here we have just written our previous expression (4.10) more globally, where Ψ_C is a function on \mathcal{M}_H and where the tensor \mathbf{J} is interpreted as a differential operator valued in $\overline{\Omega}_{\mathcal{M}_H}^1$.

One might initially be skeptical that an extension of the $\mathcal{N} = 2$ formula (4.29) exists, since the tensor \mathbf{J} is intrinsically associated to the hyperkahler structure on \mathcal{M}_H . Nonetheless, given the fiberwise nature of the computation on X , an $\mathcal{N} = 1$ analogue of \mathbf{J} does exist and is given explicitly by

$$\begin{aligned} \mathbf{J} = & \int_X d^6u (\overline{\Omega}_X)_{\bar{k}_1 \bar{k}_2 \bar{k}_3} G_{\bar{m} k_1} \delta \bar{t}^{\bar{m}} \left[G_{\bar{k}_4 k_3} \delta \bar{A}_{k_2}^a \otimes \frac{\delta}{\delta A_{\bar{k}_4}^a} + G_{\bar{k}_4 k_4} G_{\bar{k}_5 k_3} \delta \bar{U}_{k_2}^{\bar{k}_4} \otimes \frac{\delta}{\delta T_{\bar{k}_5 k_4}} + \right. \\ & \left. + G_{\bar{k}_4 k_3} G^{\bar{k}_5 k_4} \delta \bar{T}_{\bar{k}_5 k_2} \otimes \frac{\delta}{\delta U_{\bar{k}_4}^{k_4}} \right] + \dots, \\ d^6u = & du^{k_1} \wedge du^{k_2} \wedge du^{k_3} \wedge d\bar{u}^{\bar{k}_1} \wedge d\bar{u}^{\bar{k}_2} \wedge d\bar{u}^{\bar{k}_3}. \end{aligned} \quad (4.30)$$

Here Ω_X is the holomorphic three-form on X , and we introduce local holomorphic and anti-holomorphic coordinates u^k and $\bar{u}^{\bar{k}}$ on X . As in (4.17), for brevity we omit the conjugate components of \mathbf{J} , indicated by the ‘…’ above.

We have written this expression for \mathbf{J} in the same notation as for the example with $\mathcal{N} = 2$ supersymmetry to make clear that this formula (4.30) reduces to our earlier formula (4.17) when $X = E \times Y$. However, a subtle and important distinction exists between the expression in (4.30) and the expression in (4.17). In the expression above, the quantities $\delta/\delta A$, $\delta/\delta T$, and $\delta/\delta U$ must be regarded as *arbitrary* functional derivatives with respect to the gauge field A and the metric and B -field on X and not necessarily as derivatives restricted to lie along the moduli space \mathcal{M}_0 .

To explain this distinction, let us reconsider our earlier example with $\mathcal{N} = 2$ supersymmetry, for which

$$\begin{aligned} \mathbf{J} = \int_Y d^4u (\bar{\Omega}_Y)_{\bar{k}_1 \bar{k}_2} & \left[G_{\bar{k}_3 \bar{k}_2} \delta \bar{A}_{k_1}^a \otimes \frac{\delta}{\delta A_{\bar{k}_3}^a} + G_{\bar{k}_3 \bar{k}_3} G_{\bar{k}_4 \bar{k}_2} \delta \bar{U}_{k_1}^{\bar{k}_3} \otimes \frac{\delta}{\delta T_{\bar{k}_4 \bar{k}_3}} + \right. \\ & \left. + G_{\bar{k}_4 \bar{k}_2} G^{\bar{k}_3 k_4} \delta \bar{T}_{\bar{k}_3 k_1} \otimes \frac{\delta}{\delta U_{\bar{k}_4}^{k_4}} \right] + \dots \end{aligned} \quad (4.31)$$

If we wish, we can also consider $\delta/\delta A$, $\delta/\delta T$, and $\delta/\delta U$ to represent arbitrary functional derivatives in (4.31). However, because the tensors $\bar{\Omega}_Y$ and G are covariantly constant on Y and the one-forms $\delta \bar{A}$, $\delta \bar{T}$, and $\delta \bar{U}$ are harmonic on Y , the integral over Y in the definition of \mathbf{J} can only be non-zero when the functional derivatives $\delta/\delta A$, $\delta/\delta T$, and $\delta/\delta U$ are taken with respect to harmonic variations of these fields and hence lie along the hypermultiplet moduli space \mathcal{M}_H .

In contrast, in the generalized expression for \mathbf{J} in (4.30) the one-form $\delta \bar{t}$ along \mathcal{B} now also appears. Because $\delta \bar{t}$ is not covariantly constant, we find no effective projection of the functional derivatives onto the moduli space \mathcal{M}_0 .

We thus regard the generalized \mathbf{J} in (4.30) as a covariant (functional) differential operator valued in the bundle $\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^1$ on \mathcal{U} . The perturbative formula for $\varpi(t, \bar{t})$ in (4.23) is then reproduced by the action of $\mathbf{J} \circ \mathbf{J}$ on the worldvolume path integral Ψ_C (with right-moving zero-modes omitted from the integral),

$$\varpi(t, \bar{t}) = (\mathbf{J} \circ \mathbf{J})(\Psi_C), \quad (4.32)$$

where Ψ_C is again given formally by

$$\Psi_C = \exp \left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B \right) \frac{\text{Pfaff}(\bar{\partial}_{V_-})}{(\det' \bar{\partial}_{\mathcal{O}})^3 (\det' \bar{\partial}_{\mathcal{O}(-2)})}. \quad (4.33)$$

This expression for Ψ_C follows just as in Section 3.3, since the normal bundle N to C in X is still $N = \mathcal{O} \oplus \mathcal{O}(-2)$.

We now return to our claim that $\varpi(t, \bar{t})$ transforms on \mathcal{U} as a section of the bundle

$$[\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^1] \otimes [\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^1]. \quad (4.34)$$

Because \mathbf{J} is valued in $\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^1$, our claim follows from (4.32), provided that Ψ_C itself transforms as a symmetric section of $\Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1$. This fact is implicitly

guaranteed by anomaly-cancellation, since otherwise we would not eventually obtain a well-defined measure on \mathcal{B} . However, to explain this point in detail, let us consider how the various factors in the formula (4.33) for Ψ_C transform individually on \mathcal{U} . We write $\Psi_C = Z_B \cdot Z_X \cdot Z_V$, where

$$\begin{aligned} Z_B &= \exp \left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B \right), \\ Z_X &= \frac{1}{(\det' \bar{\partial}_{\mathcal{O}})^3 (\det' \bar{\partial}_{\mathcal{O}(-2)})}, \\ Z_V &= \text{Pfaff}(\bar{\partial}_{V_-}). \end{aligned} \tag{4.35}$$

We start by considering Z_X , which arises from the one-loop integral over the non-zero, left-moving bosonic modes on C . We can ignore the trivial factor $1/(\det' \bar{\partial}_{\mathcal{O}})^2$ that arises from the free bosons x^μ valued in \mathbb{C}^2 , since this factor has no interesting dependence on the moduli of X or V .

More relevant is the contribution from the bosons y^m valued in the normal bundle N to C in X . Since $N = \mathcal{O} \oplus \mathcal{O}(-2)$, this contribution is

$$\frac{1}{\det' \bar{\partial}_N} = \frac{1}{(\det' \bar{\partial}_{\mathcal{O}})(\det' \bar{\partial}_{\mathcal{O}(-2)})}. \tag{4.36}$$

This factor clearly transforms in a certain determinant line-bundle over the complex structure moduli space of X , but we must be careful to account for the fact that we project away from the zero-modes of y^m in (4.36).

To account for this projection, let us recall some basic facts about finite-dimensional determinants. If general, if we consider a linear operator $D : E \rightarrow F$ between two complex vector spaces E and F of dimension n , then the determinant of D transforms as an element of $\wedge^n E^* \otimes \wedge^n F$. This fact is manifest in the physical description of $\det(D)$ as a fermionic integral, for which we introduce fermions ψ^e transforming as coordinates on E and fermions χ_f transforming as coordinates on F^* . We then write

$$\det(D) = \int d^n \psi d^n \chi \exp(\chi_f D_e^f \psi^e). \tag{4.37}$$

Because the fermion measure $d^n \psi d^n \chi \equiv d\psi_{e_1} \cdots d\psi_{e_n} d\chi^{f_1} \cdots d\chi^{f_n}$ transforms as an element of $\wedge^n E^* \otimes \wedge^n F$, so too does $\det(D)$.

We are actually interested in the case that D has a kernel and a cokernel, so we set

$$E_0 = \text{Ker}(D), \quad F_0 = \text{Cok}(D), \tag{4.38}$$

and we assume that E has dimension m and F has dimension n . One way to define $\det'(D)$ is to introduce a metric on E and F , so that we can regard $D : E_0^\perp \rightarrow F_0^\perp$ as an operator from the orthocomplement of E_0 to the orthocomplement of F_0 . Assuming that E_0^\perp and F_0^\perp have dimension p , $\det'(D)$ is then defined as an element of $\wedge^p(E_0^\perp)^* \otimes \wedge^p F_0^\perp$.

Yet this description of $\det'(D)$ suffers from the need to introduce explicitly a projection onto E_0^\perp and F_0^\perp , and a more elegant way to describe $\det'(D)$ is suggested by the fermionic integral (4.37). We simply choose elements $(\psi_0)^{m-p} \equiv \psi_0^{e_1} \cdots \psi_0^{e_{m-p}}$ of $\wedge^{m-p} E_0$ and $(\chi^0)^{n-p} \equiv \chi_{f_1}^0 \cdots \chi_{f_{n-p}}^0$ of $\wedge^{n-p} F_0^*$, whereupon we set

$$\det'(D) = \int d^m \psi d^n \chi (\psi_0)^{m-p} (\chi^0)^{n-p} \exp(\chi_f D_e^f \psi^e). \quad (4.39)$$

Physically, the insertions of $(\psi_0)^{m-p}$ and $(\chi^0)^{n-p}$ are present to soak up the zero-modes of D , and because of these insertions we regard $\det'(D)$ as an element of $\mathcal{L}(D) \otimes \wedge^{m-p} E_0 \otimes \wedge^{n-p} F_0^*$, where we set $\mathcal{L}(D) = \wedge^m E^* \otimes \wedge^n F$.

The description of $\det'(D)$ as an element of $\mathcal{L}(D) \otimes \wedge^{\max} E_0 \otimes \wedge^{\max} F_0^*$ immediately generalizes to the infinite-dimensional case at hand. In place of D we consider the operator $\bar{\partial}_N$, and we let $\mathcal{L}(\bar{\partial}_N)$ be the associated determinant line-bundle over the complex structure moduli space of X . Similarly, by analogy to (4.38) we consider the bundles over the moduli space whose fibers are defined by

$$E_0 = \text{Ker}(\bar{\partial}_N) = H_{\bar{\partial}}^0(C, N), \quad F_0 = \text{Cok}(\bar{\partial}_N) = H_{\bar{\partial}}^1(C, N). \quad (4.40)$$

Geometrically, we identify the vector space $H_{\bar{\partial}}^0(C, N)$ as the fiber of the holomorphic tangent bundle $T\mathcal{B}$ to \mathcal{B} at the point corresponding to C , and via Serre duality we identify the vector space $H_{\bar{\partial}}^1(C, N)$ as the corresponding fiber of $\Omega_{\mathcal{B}}^1$. Combining these observations and inverting $\det'(\bar{\partial}_N)$, we conclude that Z_X transforms globally on \mathcal{U} as a section of the line-bundle

$$\pi^* \mathcal{L}^{-1}(\bar{\partial}_N) \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1. \quad (4.41)$$

We are left to consider Z_B and Z_V . As with the determinants in Z_X , the factor Z_V transforms over the complex structure moduli space of X and V as a section of the Pfaffian line-bundle $\mathcal{P}(\bar{\partial}_{V_-})$ associated to the operator $\bar{\partial}_{V_-}$. Finally, by the anomaly-cancellation mechanism we have already discussed, Z_B transforms over the complex structure moduli space as a section of $\mathcal{L}(\bar{\partial}_N) \otimes \mathcal{P}^{-1}(\bar{\partial}_{V_-})$. Hence the product $\Psi_C = Z_B \cdot Z_X \cdot Z_V$ is left to transform as a section of the bundle $\Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1$ on \mathcal{U} .

Integrating Over the Family

We are left to integrate $\varpi(t, \bar{t})$ over the parameter space \mathcal{B} of the family. As ϖ transforms on \mathcal{U} as a section of the bundle

$$\left[\overline{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^* \overline{\Omega}_{\mathcal{M}_0}^1 \right] \otimes \left[\overline{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^* \overline{\Omega}_{\mathcal{M}_0}^1 \right], \quad (4.42)$$

we only require a fiberwise metric $g_{(\mathcal{B})}$ on \mathcal{B} to define this integral. Given such a metric, we use $g_{(\mathcal{B})}$ to regard ϖ as a section of $\overline{\Omega}_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^1 \otimes \pi^* \overline{\Omega}_{\mathcal{M}_0}^1 \otimes \pi^* \overline{\Omega}_{\mathcal{M}_0}^1$, and such a section can be integrated over the fibers of \mathcal{U} to produce the required section ω of $\overline{\Omega}_{\mathcal{M}_0}^1 \otimes \overline{\Omega}_{\mathcal{M}_0}^1$ to describe the F -term.

Explicitly, in local holomorphic coordinates (t, ϕ) on \mathcal{U} , we write

$$\varpi = \varpi_{\bar{t}t\bar{i}\bar{t}\bar{j}} (d\bar{t} \wedge dt \wedge d\bar{\phi}^i) \otimes (d\bar{t} \wedge dt \wedge d\bar{\phi}^j). \quad (4.43)$$

Then in local coordinates,

$$\omega_{i\bar{j}} = \int_{\mathcal{B}} d^2 t \, g_{(\mathcal{B})}^{\bar{t}t} \varpi_{\bar{t}t\bar{i}\bar{t}\bar{j}}(t, \bar{t}). \quad (4.44)$$

Finally, the Calabi-Yau metric on X induces a natural hermitian metric on the normal bundle N on C , which in turn induces the natural hermitian metric on elements of the vector space $H_{\bar{\partial}}^0(C, N)$. Since we identify elements of $H_{\bar{\partial}}^0(C, N)$ with tangent vectors to \mathcal{B} , we have the required metric on \mathcal{B} .

Brief Remarks

Together, (4.32) and (4.44) describe in a general fashion the multi-fermion F -term generated by the one-parameter family of worldsheet instantons in X . These expressions have no reason to vanish identically in general — for instance, as we have already observed, they do not vanish in the example with $\mathcal{N} = 2$ supersymmetry! So our basic result is that worldsheet instantons can contribute to the multi-fermion F -term. As a small caveat, we have not shown directly that ω is associated to a nontrivial class in $H_{\bar{\partial}}^1(\mathcal{M}_0, T\mathcal{M}_0)$, but we believe this condition will hold in suitable examples.

It would be very interesting to have a concrete example where the deformation of \mathcal{M}_0 can be globally and explicitly determined. See [21,22,46] for some direct computations of multi-fermion F -terms in the orbifold limit of X .

As is clear from the perturbative description of ω in (4.23), the instanton-generated deformation of \mathcal{M}_0 generically mixes the complex structure and the complexified Kahler moduli of X . The perturbative moduli space \mathcal{M}_0 is not a product, since the complexified Kahler moduli fiber nontrivially over the complex structure moduli once we account for the gauge transformation of the B -field. An interesting question is whether the deformation due to instantons generically violates the fiber bundle structure of \mathcal{M}_0 that holds in worldsheet perturbation theory.

5. Other Instanton Effects

Thus far we have computed only the F -term generated by a one-parameter family of rational curves in X . More generally, we can consider a multi-parameter family of holomorphic curves of arbitrary genus in X , and we can ask what F -term is generated by this family.

To answer this question, we now extend the instanton computations in Section 3 and Section 4 in two ways. First, we compute the F -term generated by an arbitrary multi-parameter family of rational curves in X . Second, we compute the F -term generated by an isolated holomorphic curve of arbitrary genus in X . The general case is then midway between these extreme cases.

5.1. Multi-Fermion F -terms From Multi-Parameter Families

We start by considering the F -term generated by a multi-parameter family of rational curves in X , since the analysis in this case is an immediate generalization of the analysis we just performed in the case of a one-parameter family. We show that a family of complex dimension p naturally generates the corresponding multi-fermion F -term of degree p in (2.7).

For simplicity, we assume that the generic curve C in the family has no obstructed deformations. We thus model the local geometry of C in X on the normal bundle $N = \mathcal{O}(p-1) \oplus \mathcal{O}(-p-1)$, and since $H_{\bar{\partial}}^0(C, N)$ has complex dimension p , the parameter space \mathcal{B} for the family also has dimension p .

With these assumptions, the right-moving worldvolume fermions $\bar{\chi}_{\dot{\alpha}}^{\bar{m}}$ which are valued in the bundle \bar{N} have $2p$ zero-modes on C . These fermion zero-modes must be absorbed in a perturbative instanton computation using the same worldvolume interactions (4.22) that we used to generate the multi-fermion F -term in the special case $p=1$. Because we now have $2p$ fermion zero-modes, we need $2p$ interaction terms to soak up the zero-modes, and hence in worldvolume perturbation theory we generate an expression which immediately generalizes (4.23) and now involves a correlator of $2p$ left-moving currents on C . We will not write this perturbative expression here, since it is both unwieldy and un-illuminating.

Nevertheless we will present an equivalent expression for the multi-fermion F -term using the more compact formalism of Section 4.2. In this formalism, each interaction term that we bring down from the worldvolume theory on C corresponds to the action of the

differential operator \mathbf{J} in (4.30) on the path integral Ψ_C (with right-moving zero-modes omitted),

$$\Psi_C = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B\right) \frac{\text{Pfaff}(\bar{\partial}_{V_-})}{(\det' \bar{\partial}_O)^2 (\det' \bar{\partial}_N)}. \quad (5.1)$$

So schematically, the instanton contribution from C is given by a formula generalizing (4.32),

$$\varpi(t_1, \dots, t_p, \bar{t}_1, \dots, \bar{t}_p) = (\mathbf{J} \circ \dots \circ \mathbf{J})(\Psi_C), \quad (5.2)$$

where the operator \mathbf{J} acts $2p$ times in (5.2), and where t_1, \dots, t_p and $\bar{t}_1, \dots, \bar{t}_p$ are local holomorphic and anti-holomorphic coordinates on the parameter space \mathcal{B} .

Exactly as in our discussion in the case $p = 1$, the quantity Ψ_C transforms as a symmetric section of the bundle $\Omega_{\mathcal{U}/\mathcal{M}_0}^p \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^p$ on \mathcal{U} , due to the appearance of $\det' \bar{\partial}_N$ in (5.1), and the differential operator $\mathbf{J}^{2p} = (\mathbf{J} \circ \dots \circ \mathbf{J})$ is valued in the bundle

$$[\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^p \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^p] \otimes [\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^p \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^p]. \quad (5.3)$$

Hence $\varpi(t_1, \dots, t_p, \bar{t}_1, \dots, \bar{t}_p)$ transforms on \mathcal{U} as a section of the bundle

$$[\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^p \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^p \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^p] \otimes [\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^p \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^p \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^p]. \quad (5.4)$$

To integrate this quantity over \mathcal{B} , we again use the natural metric on \mathcal{B} to contract one set of indices in $\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^p$ and $\Omega_{\mathcal{U}/\mathcal{M}_0}^p$ so as to regard ϖ as a section of $\bar{\Omega}_{\mathcal{U}/\mathcal{M}_0}^p \otimes \Omega_{\mathcal{U}/\mathcal{M}_0}^p \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^p \otimes \pi^* \bar{\Omega}_{\mathcal{M}_0}^p$. We then integrate ϖ fiberwise to produce the section ω of $\bar{\Omega}_{\mathcal{M}_0}^p \otimes \bar{\Omega}_{\mathcal{M}_0}^p$ that specifies the multi-fermion F -term of degree p .

5.2. Higher-Derivative F -terms From Higher Genus Curves

We now consider the case that C is an isolated holomorphic curve of genus $g \geq 1$ in X . Again, we wish to compute the F -term generated by C .

Although instanton effects due to higher genus curves have not been much considered in the context of heterotic Calabi-Yau compactification, they have been extensively studied in the context of Type IIA Calabi-Yau compactification, where they are computed by the A -model topological string. In this case, C generates a well-known, $\mathcal{N} = 2$ supersymmetric chiral interaction which is proportional to $(\mathcal{W}^2)^g$, where $\mathcal{W}^2 \equiv \mathcal{W}_{\alpha\beta}\mathcal{W}^{\alpha\beta}$ and $\mathcal{W}_{\alpha\beta}$ denotes the superfield containing the self-dual Weyl multiplet of $\mathcal{N} = 2$ supergravity. (The lowest component of $\mathcal{W}_{\alpha\beta}$ is the self-dual component of the graviphoton field strength.) This fact was originally derived in [13,14] using the RNS formalism and was also discussed in the

hybrid formalism in [15]. As a warmup for the higher genus heterotic computation, we first provide a simple derivation of this classic result via the physical gauge formalism.

An A-Model Instanton Computation in Physical Gauge

To set up the *A*-model instanton computation, we first describe the physical degrees of freedom on a Type IIA fundamental string worldsheet wrapping C , as we did for the heterotic string in Section 3.1. Of course, the worldvolume bosons on C are exactly as for the heterotic string and are valued in the bundle $\mathcal{O}^2 \oplus N$.

As for the worldvolume fermions on C , these degrees of freedom are also determined by our previous discussion of the heterotic string. The right-moving fermions on C are precisely the same as for the heterotic string, and the left-moving fermions are then determined by the fact that the worldvolume theory on C is non-chiral. Thus, as in Section 3.1, the right-moving fermions on C transform as sections of the bundles

$$S_+(\mathcal{O}^2) \otimes \overline{N}, \quad S_-(\mathcal{O}^2) \otimes \overline{\mathcal{O}}, \quad S_-(\mathcal{O}^2) \otimes \overline{\Omega}_C^1, \quad (5.5)$$

and the left-moving fermions on C transform in the conjugate bundles,

$$S_+(\mathcal{O}^2) \otimes N, \quad S_-(\mathcal{O}^2) \otimes \mathcal{O}, \quad S_-(\mathcal{O}^2) \otimes \Omega_C^1. \quad (5.6)$$

We introduce the following notation for these fermions,

$$\begin{aligned} \bar{\chi}_{\dot{\alpha}}^m &\in \Gamma(C, S_+(\mathcal{O}^2) \otimes N), & \bar{\chi}_{\dot{\alpha}}^{\overline{m}} &\in \Gamma(C, S_+(\mathcal{O}^2) \otimes \overline{N}), \\ \theta_L^\alpha &\in \Gamma(C, S_-(\mathcal{O}^2) \otimes \mathcal{O}), & \theta_R^\alpha &\in \Gamma(C, S_-(\mathcal{O}^2) \otimes \overline{\mathcal{O}}), \\ \theta_z^\alpha &\in \Gamma(C, S_-(\mathcal{O}^2) \otimes \Omega_C^1), & \theta_{\overline{z}}^\alpha &\in \Gamma(C, S_-(\mathcal{O}^2) \otimes \overline{\Omega}_C^1). \end{aligned} \quad (5.7)$$

From (5.7) we can immediately count fermion zero-modes on C . First, we see that C always carries four fermion zero-modes arising from θ_L^α and θ_R^α . These zero-modes are trivially associated to the four supersymmetries broken by C , and in combination with the four bosonic zero-modes associated to translation in \mathbb{C}^2 , these zero-modes generate the $\mathcal{N} = 2$ chiral measure $d^4x d^4\theta$. Unlike our previous computation, we assume for simplicity that C is isolated in X , so that the fermions $\bar{\chi}_{\dot{\alpha}}^m$ and $\bar{\chi}_{\dot{\alpha}}^{\overline{m}}$ have no zero-modes. However, if C has genus $g \geq 1$, then the fermions θ_z^α and $\theta_{\overline{z}}^\alpha$ carry $4g$ additional zero-modes associated to the g holomorphic sections of Ω_C^1 . We are most interested in the effect of the extra zero-modes of θ_z^α and $\theta_{\overline{z}}^\alpha$ on the instanton computation, since these zero-modes control the structure of the chiral interaction generated by C .

As is convenient in the A -model, let us combine the fermions θ_z^α and $\theta_{\bar{z}}^\alpha$, which transform respectively as one-forms of type $(1, 0)$ and type $(0, 1)$ on C , into a single fermion ρ^α which transforms as a complex one-form of arbitrary type on C ,

$$\rho^\alpha = \theta_z^\alpha dz + \theta_{\bar{z}}^\alpha d\bar{z}. \quad (5.8)$$

As usual, Hodge theory on C identifies the zero-modes of ρ^α with elements of the de Rham cohomology group $H^1(C, \mathbb{C})$, and to define the fermionic measure $d^{4g}\rho$ on the $4g$ zero-modes of ρ^α we require a natural measure on the complex vector space $H \equiv H^1(C, \mathbb{C})$.

Such a measure arises from the intersection pairing on C , which induces a natural symplectic form on $H^1(C, \mathbb{C})$. Explicitly, if η and ξ are any two classes in $H^1(C, \mathbb{C})$, the symplectic form Ω_H is given by

$$\Omega_H(\eta, \xi) = \int_C \eta \wedge \xi. \quad (5.9)$$

We now use the top-form $\Omega_H^g/g!$ to define the fermionic measure $d^{4g}\rho$. One very important fact about this measure is that it is manifestly independent of the complex structure on C and hence is independent of the complex structure on X , as we expect for the A -model.

To evaluate the chiral correction δS generated by C , we again compute the partition function on C , modulo perturbative corrections in α' . Because the worldvolume theory on C preserves four supercharges, the one-loop Gaussian integrals over the non-zero modes of the bosons and the fermions on C now cancel, and the only non-trivial integral to perform is the integral over the $4g$ zero-modes of ρ^α .

To perform this fermionic integral, we once again bring down interaction terms from the worldvolume theory on C which couple ρ^α to the supergravity background. As in Section 4.2, the easiest way to deduce these interactions is simply to make the natural guess, which can then be checked directly using the results of [44]. In the case at hand, the required worldvolume interactions take the extremely simple form,

$$I_C^{\text{int}} = \int_C \mathcal{W}_{\alpha\beta} \rho^\alpha \wedge \rho^\beta. \quad (5.10)$$

As a little check, we note that since ρ^α is fermionic and a one-form, the two-form $\rho^\alpha \wedge \rho^\beta$ is necessarily symmetric in the spinor indices α and β and hence does couple to the self-dual component of the Weyl multiplet $\mathcal{W}_{\alpha\beta}$.

Thus, to absorb the $4g$ zero-modes of ρ^α , we bring down $2g$ insertions of the interaction term in (5.10). Suppressing numerical factors, we directly compute

$$\begin{aligned}\delta S &= \int d^4x d^4\theta d^{4g}\rho \left[\int_C \mathcal{W}_{\alpha\beta} \rho^\alpha \wedge \rho^\beta \right]^{2g} \exp \left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B \right), \\ &= \int d^4x d^4\theta (\mathcal{W}^2)^g \exp \left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B \right).\end{aligned}\tag{5.11}$$

In simplifying the first expression in (5.11), we recognize that the interaction term involves the same symplectic pairing in (5.9) that we use to define the zero-mode measure $d^{4g}\rho$. Consequently the zero-mode integral is immediate. This expression for δS is the usual result for instanton contributions in the A -model, for which the sum over holomorphic curves C in X , weighted by the standard exponential factor in (5.11) with a suitable prefactor for multiple-covers, defines the A -model partition function F_g at genus g .

Heterotic Generalization

We now compute the corresponding F -term generated by a heterotic worldsheet instanton wrapping C . This computation has previously been sketched in the RNS formalism by Antoniadis and collaborators [22].

As for the A -model computation, we assume that C is isolated in X , so that the only right-moving fermion zero-modes on C are the two zero-modes of θ^α which appear in the chiral measure $d^4x d^2\theta$ and the $2g$ zero-modes of $\theta_{\bar{z}}^\alpha$, which arise exactly as in the A -model. The zero-modes of $\theta_{\bar{z}}^\alpha$ control the structure of the F -term generated by the instanton, so we begin by describing the fermionic zero-mode measure $d^{2g}\theta_{\bar{z}}$. This discussion runs closely parallel to our discussion of the integral over \mathcal{B} in Section 4.2.

As in Section 4.2, to discuss the fermionic zero-mode measure we introduce vector spaces E_0 and F_0 describing the kernel and cokernel of the $\bar{\partial}$ operator coupled to the trivial bundle \mathcal{O}^2 on C ,

$$E_0 = \text{Ker}(\bar{\partial}_{\mathcal{O}^2}) = H_{\bar{\partial}}^0(C, \mathcal{O}^2), \quad F_0 = \text{Cok}(\bar{\partial}_{\mathcal{O}^2}) = H_{\bar{\partial}}^1(C, \mathcal{O}^2).\tag{5.12}$$

As the complex structure of X varies, E_0 and F_0 generally fiber over the moduli space as holomorphic bundles, and with a slight abuse of notation we also identify E_0 and F_0 with the corresponding bundles. The trivial zero-modes of θ^α then transform as sections of \overline{E}_0 , and via Serre duality the zero-modes of $\theta_{\bar{z}}^\alpha$ transform as sections of \overline{F}_0^* . Thus, the fermionic integral with respect to $d^2\theta d^{2g}\theta_{\bar{z}}$ should be interpreted globally as producing a section of $\wedge^2 \overline{E}_0^* \otimes \wedge^{2g} \overline{F}_0$.

As we recall momentarily, a natural hermitian metric, depending on the complex structure of C , exists on F_0 . Hence the fermionic integral with respect to $d^2\theta d^{2g}\theta_{\bar{z}}$ is well-defined provided that the integrand itself transforms as a section of $\wedge^2 E_0^* \otimes \wedge^{2g} F_0$. In this case, we use the metric on F_0 to regard the resulting section of $\wedge^2 E_0^* \otimes \wedge^2 \overline{E}_0^* \otimes \wedge^{2g} F_0 \otimes \wedge^{2g} \overline{F}_0$ as a section of $\wedge^2 E_0^* \otimes \wedge^2 \overline{E}_0^*$, which then represents the measure d^4x .

Exactly as in Section 4.2, this section of $\wedge^2 E_0^* \otimes \wedge^{2g} F_0$ arises from the one-loop integral over the non-zero, left-moving modes of the bosons x^μ which are also valued in \mathcal{O}^2 . The one-loop integral over these modes produces the familiar factor $1/(\det' \overline{\partial}_{\mathcal{O}})^2$, which by our discussion in Section 4.2 transforms as a section of $\mathcal{L}^{-1}(\overline{\partial}_{\mathcal{O}^2}) \otimes \wedge^2 E_0^* \otimes \wedge^{2g} F_0$ over the complex structure moduli space of X . The determinant line-bundle $\mathcal{L}^{-1}(\overline{\partial}_{\mathcal{O}^2})$ is cancelled by the classical factor $\exp(i \int_C B)$ as before to produce the required section of $\wedge^2 E_0^* \otimes \wedge^{2g} F_0$.

As for the metric on F_0 , we use the standard metric that arises from the period matrix of C . To describe this metric, we choose a symplectic basis A_i and B_i for $i = 1, \dots, g$ for the usual A - and B -cycles on C , and we let γ_i be a normalized basis for $H_{\overline{\partial}}^0(C, \Omega_C^1)$. We note that F_0^* is the direct sum of two copies of $H_{\overline{\partial}}^0(C, \Omega_C^1)$, and the normalization condition on the γ_i is simply the condition that

$$\int_{A_i} \gamma_j = \delta_{ij}. \quad (5.13)$$

We introduce the usual period matrix τ , whose elements are given by

$$\tau_{ij} = \int_{B_i} \gamma_j. \quad (5.14)$$

As explained in Chapter 2 of [47], the Riemann bilinear relations imply that τ is symmetric, and we define a hermitian metric on the space $H_{\overline{\partial}}^0(C, \Omega_C^1)$ by

$$(\gamma_i, \gamma_j) = \frac{i}{2} \int_C \overline{\gamma_i} \wedge \gamma_j = \frac{i}{2} \overline{\int_{B_j} \gamma_i} - \frac{i}{2} \int_{B_i} \gamma_j = \text{Im } \tau_{ij}. \quad (5.15)$$

The hermitian metric (5.15) on $H_{\overline{\partial}}^0(C, \Omega_C^1)$ immediately induces a corresponding metric on F_0 .

We are now prepared to compute the F -term generated by a heterotic worldsheet instanton wrapping C in X . To soak up the zero-modes of the fermion $\theta_{\bar{z}}^\alpha$, we must again consider interaction terms on C which couple $\theta_{\bar{z}}^\alpha$ to the supergravity background. We can

easily guess such a coupling, which is the natural heterotic generalization of the Type IIA coupling in (5.10),

$$I_C^{\text{int}} = \int_C d^2z \text{Tr}(W_\alpha j_z) \theta_{\bar{z}}^\alpha. \quad (5.16)$$

Here W_α is the usual $\mathcal{N} = 1$ chiral gauge superfield whose lowest component is the four-dimension gaugino. As with (5.10), one can directly check using the results in [42,43] that this coupling is indeed present on the heterotic string worldvolume.

To compute the F -term generated by C , we evaluate the worldvolume partition function with $2g$ insertions of the interaction term in (5.16) to absorb the zero-modes of $\theta_{\bar{z}}^\alpha$. We thus find a correction to the effective action which again involves a correlator of left-moving currents on C of the form

$$\begin{aligned} \delta S &= \int d^4x d^2\theta d^{2g}\theta_{\bar{z}} \left\langle \left[\int_C d^2z \text{Tr}(W_\alpha j_z) \theta_{\bar{z}}^\alpha \right]^{2g} \right\rangle', \\ &= \int d^4x d^2\theta (W^2)^{a_1 b_1} \cdots (W^2)^{a_g b_g} \text{Im} \tau^{i_1 j_1} \cdots \text{Im} \tau^{i_g j_g} \text{Im} \tau^{k_1 l_1} \cdots \text{Im} \tau^{k_g l_g} \times \\ &\quad \times \int_{C^{2g}} d^2z_1 d^2w_1 \cdots d^2z_g d^2w_g (\bar{\gamma}_{i_1})_{\bar{z}_1} \cdots (\bar{\gamma}_{i_g})_{\bar{z}_g} (\bar{\gamma}_{k_1})_{\bar{w}_1} \cdots (\bar{\gamma}_{k_g})_{\bar{w}_g} \times \\ &\quad \times \langle j_{z_1 a_1} j_{w_1 b_1} \cdots j_{z_g a_g} j_{w_g b_g} \rangle' (\gamma_{j_1} \wedge \cdots \wedge \gamma_{j_g} \otimes \gamma_{l_1} \wedge \cdots \wedge \gamma_{l_g}). \end{aligned} \quad (5.17)$$

Here $(W^2)^{ab} \equiv W_\alpha^a W^{\alpha b}$, where a and b are adjoint-valued gauge indices. Precisely as for the multi-fermion F -terms in Section 2, the expression $(W^2)^{a_1 b_1} \cdots (W^2)^{a_g b_g}$ is anti-symmetric on each set of a and b indices and is symmetric under the exchange of a_i and b_i .

As is apparent in (5.17), we use the period matrix τ of C to trivialize the section of $\wedge^{2g} F_0 \otimes \wedge^{2g} \bar{F}_0$ that implicitly arises once we perform the integral with respect to $d^{2g}\theta_{\bar{z}}$. In particular, to make the dependence of the correlator $\langle \cdots \rangle'$ on our choice of basis γ_i explicit, we write this correlator as a (linear) function of the section $\gamma_{j_1} \wedge \cdots \wedge \gamma_{j_g} \otimes \gamma_{l_1} \wedge \cdots \wedge \gamma_{l_g}$ in (5.17).

This expression for δS does not vanish identically, and we can simplify it slightly in the following special case. We consider a tensor component of $(W^2)^{a_1 b_1} \cdots (W^2)^{a_g b_g}$ for which all the gauge indices a and b lie in a Cartan subalgebra of the gauge group and for which all these indices are distinct. Since the (suitably normalized) correlator of any two currents on C takes the local form

$$j_z^a(z) j_w^b(0) = \frac{k \delta^{ab}}{z^2} + \frac{i f_c^{ab} j^c(0)}{z} + \text{regular}, \quad (5.18)$$

for some level k and structure constants f_c^{ab} , our assumptions imply that the correlator of currents in (5.17) is regular on the product C^{2g} . Since this correlator must also be holomorphic on C^{2g} , we deduce that it can be evaluated by reducing to the zero-modes of the currents, so that we substitute for j_{za} as

$$j_{za} = Q_a^i (\gamma_i)_z. \quad (5.19)$$

Under this substitution, $\oint_{A_i} j_{za} = Q_a^i$, from which we see that Q_a^i is the abelian charge operator indexed by a and associated to the cycle A_i on C . Substituting (5.19) into (5.17), we produce factors of τ to cancel those already appearing in (5.17), and δS simplifies to

$$\begin{aligned} \delta S = & \int d^4x d^2\theta (W^2)^{\hat{a}_1 \hat{b}_1} \dots (W^2)^{\hat{a}_g \hat{b}_g} \times \\ & \times \left\langle Q_{\hat{a}_1}^{i_1} \dots Q_{\hat{a}_g}^{i_g} Q_{\hat{b}_1}^{j_1} \dots Q_{\hat{b}_g}^{j_g} \right\rangle' (\gamma_{i_1} \wedge \dots \wedge \gamma_{i_g} \otimes \gamma_{j_1} \wedge \dots \wedge \gamma_{j_g}). \end{aligned} \quad (5.20)$$

The hats on the indices \hat{a} and \hat{b} in (5.20) just serve to remind us that we only evaluate δS for the special components of $(W^2)^{a_1 b_1} \dots (W^2)^{a_g b_g}$ described above.

Open String Worldsheet Instantons in the A-Model

As a final example, we now apply the physical gauge formalism to perform worldsheet instanton computations in the open string A -model. Thus, we consider Type IIA string theory compactified on a Calabi-Yau threefold X containing a special Lagrangian cycle L . On L we wrap N $D6$ -branes, and we suppose that C is a holomorphic curve with g handles and h holes, the boundaries of which end on L . As before, we assume that C is isolated as a holomorphic curve in X . Instanton computations for such curves have been performed most recently in the RNS formalism by Antoniadis and collaborators in [23] and also computed earlier in the hybrid formalism in [17–20].

Let us write the boundary of C as a disjoint union of circles Γ_k for $k = 1, \dots, h$,

$$\partial C = \bigsqcup_{k=1}^h \Gamma_k. \quad (5.21)$$

On the boundary ∂C , the worldvolume bosons and fermions satisfy the usual Dirichlet and Neumann boundary conditions to describe a $D6$ -brane wrapping L . In particular, the worldvolume fermions with zero-modes are those related by the unbroken supercharges to the Neumann directions in \mathbb{C}^2 parametrized by x^μ , and hence these fermions satisfy the boundary conditions

$$\theta_L^\alpha = \theta_R^\alpha \Big|_{\partial C}, \quad \theta_z^\alpha = \theta_{\bar{z}}^\alpha \Big|_{\partial C}. \quad (5.22)$$

As previously, we set $\rho^\alpha = \theta_z^\alpha dz + \theta_{\bar{z}}^\alpha d\bar{z}$.

To describe the fermion zero-modes satisfying (5.22) on C , we apply the usual “doubling trick” (or the Schwarz reflection principle) to regard C as the quotient by an anti-holomorphic involution of a closed curve \tilde{C} of genus $\tilde{g} = 2g + h - 1$, so that $C = \tilde{C}/\mathbb{Z}_2$. Locally this involution sends $z \mapsto w = \bar{z}$, and under this involution the fixed points of \tilde{C} become the boundaries Γ_k of C .

With \tilde{C} in hand, the fermion zero-modes which satisfy (5.22) can then be described as the elements of the de Rham cohomology groups $H^0(\tilde{C}, \mathbb{C})$ and $H^1(\tilde{C}, \mathbb{C})$ which are invariant under the involution and hence descend to C . This condition is trivial on the constants in $H^0(\tilde{C}, \mathbb{C})$, so that we find two zero-modes from the fermions θ_R^α and θ_L^α . These zero-modes, along with the bosonic zero-modes of x^μ , now generate the $\mathcal{N} = 1$ chiral measure $d^4x d^2\theta$.

On the other hand, the condition that an element of $H^1(\tilde{C}, \mathbb{C})$ be even under the involution is nontrivial, and the invariant subspace $H_{\text{ev}}^1(\tilde{C}, \mathbb{C})$ has complex dimension $\tilde{g} = 2g + h - 1$. Elements of this subspace are specified by their periods around the A - and B -cycles of the original curve C as well as their periods around any $h - 1$ of the boundary components Γ_k of C . Of course, in the homology of C we have a relation

$$\sum_{k=1}^h [\Gamma_k] = 0, \quad (5.23)$$

which obviates the need to specify a period about the remaining boundary component of C .

As in our other examples, the first step in performing the instanton computation on C is to discuss the zero-mode measure $d^{2(2g+h-1)}\rho$. In the earlier case for which C had no boundaries, we defined this measure using the natural symplectic form (5.9) associated to the intersection pairing in $H^1(C, \mathbb{C})$. We can use the same intersection pairing to define a measure on the $2g$ dimensional subspace of $H_{\text{ev}}^1(\tilde{C}, \mathbb{C})$ associated to periods of ρ^α on the A - and B -cycles of C , but this intersection pairing is degenerate along the $h - 1$ dimensional subspace of $H_{\text{ev}}^1(\tilde{C}, \mathbb{C})$ associated to the periods of ρ^α along the boundary of C .

To define the fermionic measure along the subspace of $H_{\text{ev}}^1(\tilde{C}, \mathbb{C})$ associated to the boundary periods of ρ^α , we note that each boundary component Γ_k , which lifts to a one-cycle in \tilde{C} , defines a one-form on $H_{\text{ev}}^1(\tilde{C}, \mathbb{C})$ via the canonical pairing $\eta \mapsto \int_{\Gamma_k} \eta$ for η in $H_{\text{ev}}^1(\tilde{C}, \mathbb{C})$. Naively, we would like to define the required measure by wedging together the one-forms associated to each boundary component Γ_k for $k = 1, \dots, h$, but we must

take into account the homology relation (5.23). To account for this relation, we find it convenient to first regard the periods of ρ^α on the boundaries Γ_k as unconstrained and then to define the measure $d^{2(2g+h-1)}\rho$ using an explicit fermionic delta-function to enforce the constraint implied by (5.23). This fermionic delta-function is given by

$$\delta^{(2)}\left(\sum_{k=1}^h \int_{\Gamma_k} \rho\right) = \left(\sum_{k=1}^h \int_{\Gamma_k} \rho^\alpha\right) \cdot \left(\sum_{l=1}^h \int_{\Gamma_l} \rho_\alpha\right). \quad (5.24)$$

So we write

$$d^{2(2g+h-1)}\rho \equiv d^{2(2g+h)}\rho \cdot \delta^{(2)}\left(\sum_{k=1}^h \int_{\Gamma_k} \rho\right), \quad (5.25)$$

where $d^{2(2g+h)}\rho$ is the measure defined using the symplectic pairing on C associated to its A - and B -cycles as well as the (formal) differential form of degree h associated to the boundary components Γ_k of C .

We now compute exactly as in the closed-string A -model. Besides the bulk interactions involving the Weyl superfield $\mathcal{W}_{\alpha\beta}$ in (5.10), we must also consider boundary interactions to soak up the fermion zero-modes associated to the periods of ρ^α around Γ_k . These boundary interactions take the usual form of holonomy operators,

$$I_{\partial C}^{\text{int}} = \prod_{k=1}^h \text{Tr} \left[P \exp \left(\oint_{\Gamma_k} W_\alpha \rho^\alpha \right) \right], \quad (5.26)$$

where W_α is the $\mathcal{N} = 1$ gaugino superfield that appeared in the previous heterotic computation.

As explained by Dijkgraaf and collaborators in [20], somewhat more complicated bulk couplings to the gravitino field strength $E_{\alpha\beta\gamma}$ also turn out to be relevant to this computation, but for simplicity we set $E_{\alpha\beta\gamma} \equiv 0$. A computation in the physical gauge formalism with non-zero $E_{\alpha\beta\gamma}$ would proceed in direct analogy to the hybrid computation in [20].

As in the case of the closed string A -model, the worldvolume path integral on C reduces to an integral over the zero-modes of ρ^α which we immediately evaluate,

$$\begin{aligned} \delta S &= \int d^4x d^2\theta d^{2(2g+h)}\rho \delta^{(2)}\left(\sum_{k=1}^h \int_{\Gamma_k} \rho\right) \prod_{k=1}^h \text{Tr} \left[P \exp \left(\int_{\Gamma_k} W_\alpha \rho^\alpha \right) \right] \times \\ &\quad \times \left[\int_C \mathcal{W}_{\alpha\beta} \rho^\alpha \wedge \rho^\beta \right]^{2g} \exp \left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B \right), \\ &= \int d^4x d^2\theta \left[Nh (\text{Tr}(W_\alpha W^\alpha))^{h-1} + \binom{h}{2} \text{Tr}(W_\alpha) \text{Tr}(W^\alpha) (\text{Tr}(W_\beta W^\beta))^{h-2} \right] \times \\ &\quad \times (\mathcal{W}_{\alpha\beta} \mathcal{W}^{\alpha\beta})^g \exp \left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B \right). \end{aligned} \quad (5.27)$$

The two summands in (5.27), respectively proportional to Nh and $\binom{h}{2}$, arise from two qualitatively distinct ways of distributing the $2(h - 1)$ boundary zero-modes of ρ^α . Either the boundary zero-modes of ρ^α can be paired at any $h - 1$ of the boundary components of C , leaving one boundary with no zero-mode insertions, or $h - 2$ boundary components can carry two zero-modes with the remaining two boundary components carrying one zero-mode apiece. In the first case, the combinatorial factor h arises from the choice of the distinguished boundary component with no zero-mode insertions, and the factor N arises from the trace over gauge indices on this boundary. In the second case, the combinatorial factor $\binom{h}{2}$ arises from the pair of distinguished boundary components, and we have no trace over gauge indices. This expression (5.27) for δS reproduces the basic expectations of [17] concerning these open string instanton contributions in the A -model.

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